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MODELS FOR THE EXPANSION PLANNING OF A MULTI-PLANT, MULTI-PRODUCT FIRM

A Thesis

Presented to

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by

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MODELS FOR THE EXPANSION PLANNING OF A

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SUMMARY

The general purpose of this study is to present a quantitative method of solution to the expansion planning problem. This problem appears at the time of planning capital investments (or disinvestments) to meet anticipated changes in demand. It is assumed that the production-consumer system is made up of several manufacturing plants which could ship their outputs simultaneously to different customers. Thus, there exists a strong relationship between plants and their markets. Some plants may send semifinished products to other plants, as a way to increase the production capacity in a department.

Because of the fact that optimization of the size of each plant does not optimize the production capacity of the system, it is necessary to consider all plants (actual and proposed) and their markets together. In this way, it is possible to ascertain if the increase in capacity of one plant may make unnecessary an extra investment in another plant of the system.

In accordance with particular situations, four models are presented, each one having different assumptions. Model A, being the simplest, introduces the basic concepts in the expansion of capacity and considers only one period. Model B represents the situation when the planning horizon is divided into several periods. Model C extends Model B by allowing production for inventory as a way to reduce extra investments. Model D shows the case in which new investments change

the variable productions cost of a plant, i.e., when economies or diseconomies of scale are present in new investments.

All models fit the framework of mixed-integer programming problems. Some algorithms are discussed. It was found that Branch and Bound methods have had good results when just a few integer variables exist, as in this case. In particular, the algorithm developed by Rebelin was proposed to solve the models described in this work.

In order to show the applicability of the models and their solutions, a practical case was modeled and solved. The real problem fits the conditions of Model B. The system contains seven plants and nine regions. Although each plant may have four departments, only their assembly department was included in the example, on the basis that it was common for all plants. Each assembly department has a number of lines, each one having the capability to produce one or several products.

Initially, it was found that there were more than 30 different products. However, some of these differences were not significant for this study. Therefore, 12 representative products were defined in order to reduce the dimensionality of the problem.

The variables in the model were divided in three groups, one for each period of time of the planning horizons. Each period has 70 constraints (18 relating to production facilities and 52 relating to markets regions) and 135 variables.

A variable gives the name of the product, where it is produced (plant and line), and the market where it is shipped.

Initially eight alternatives (zero-one variables) were proposed to increase or decrease the production capacity of the system and satisfy the forecast demand. One the optimal mixed solution was obtained, a new set of alternatives was proposed.

The program was run on a Burroughs 5500 computer, using remote terminals. Each trial took around ten minutes, but it can reduced easily by storing the optimal continuous solution.

The solution of this example gave two types of solution. One involves long-term planning, that is, the selection of investments (either decreases or increases in production capacity) for the next planning horizon. The other type of solution concerns medium-range planning, that is, which plant should ship a given product to a certain customer in each period.

Some extensions of this work might consist of the following:

a) Separating total cost into three parts, namely, fixed-fixed, fixed-variable, and variable cost.

b) Including a criterion of minimum rate of return on investment.

c) Developing a methodology to include intangibles and to generate investment alternatives.

d) Considering the forecast demand as a probability distribution rather than as certain data.

CHAPTER I

INTRODUCTION

1.1 Background

The purpose of this study is to present a quantitative method for solving the expansion planning problem of an operating firm. It is assumed that the firm consists of more than one plant, manufactures many products, and has customers located at many sites. The expansion problem appears whenever management is planning capital investments to meet anticipated demands. Typical examples of the planning decisions that management must make are the following:

a) when and where to allocate the capacity increases or decreases;

b) whether or not a new plant should be built and, if so, where;

c) whether or not an existing plant should be partially or totally closed and, if so, when; and

d) whether or not the firm should centralize or decentralize its operations.

In a firm with only one plant, or with a few plants scattered around a region so that there is no interaction among them, decisions can be made on the basis of a single plant. One method widely used is to calculate the figure of merit (profit or cost) for each alternative and rank them in decreasing order of priority until some scarce resource, e.g., capital, is depleted. In efforts to use this approach in problems involving multi-plant firms, independence between plants or projects is often assumed in order to facilitate computation, even though the real situation indicates that strong interactions exist between plants. In such a situation, the method will most likely lead to a suboptimization. For example, expansion of one plant in the system could make the construction of a new one unnecessary, or vice versa, although an independent analysis indicates that both are equally good investments. Therefore, the method of solution of the expansion problem should take into account existing interrelationships between plants, their resources, their markets, and the scheduling of alternatives over the planning horizon.

The method also should permit the analysis of a decision, so that management may evaluate some alternatives before the decision is implemented in the real world.

1.2 The Problem

The initial idea for this research was suggested by officials of the American Can Company, but the results of the study are useful in the formulation and solution of expansion planning problems in many multi-plant, multi-product firms. In this presentation, the problems, the models, and the solution procedure first are described in detail in general terms. There a particular case is presented in order to show the application of a model and its solution.

Assume first that a firm has many plants operating in a given region. Each plant is able to satisfy the final demand of the customers and/or the demand for intermediate products requested by other plants. For a projected increase or decrease in the demand for each final product, the management of the firm wants a production plan including when and where to allocate capacity increases such that the total profit is a maximum over a specified planning horizon.

The capacity of a plant or a department is measured by the number of lines or pieces of equipment. In general, it is expressed in discrete units. For example, the capacity of an assembly department is specified by the number of assembly lines. It is assumed that capacity increases are made by adding assembly lines in discrete units.

The problem of allocation of increases (or decreases) in productive capacity requires the study of such factors as demand; price of products; requirements and cost of raw materials and labor for each period considered; number, size and cost of facilities; capacity for different production processes; timing of the installations; location of the plant and cost of transportation; and availability and cost of capital funds.

A naive attempt to solve this problem would be to isolate each plant and try to optimize its size. However, optimizing the size separately does not necessarily give the maximum profit for the system. This approach may result in overestimating the investment required for each plant. On the other hand, if the system of plants is examined as a whole, management may obtain a solution near to the optimum. (The system is defined as the set of plants, customers, suppliers of raw materials, and the interrelationships between them). In this case, for example, the excess capacity in one plant may be used to satisfy the shortage of a nearby plant, during a certain period of time, at a lower cost than can be achieved by extra investment.

In the preceding discussion, it was assumed that a production-consumption system is already in operation. The optimal configuration of a nonexistent new system is not studied here. This distinction is important for two reasons. The first reason is that with a system already in operation, the decision maker is concerned only with incremental costs. That is, the cost of operating the present system is fixed. By specifying increases or decreases in plant capacities, the system is modified and only the extra cost should be considered. The second reason is that in a new system an extra dimension, the location of the plants, is included. In the system already in operation this variable is fixed. In the set of possible alternatives, to be described later, new plants may be included, but their locations, which depend upon the existing plants, are specified in advance.

Two types of investment decisions to increase the capacity of the system are generally made. The first occurs when there exist economies (or diseconomies) of scale in the physical plant. Thus, an increase in capacity will reduce (increase) the unit variable cost. In this case, the decision maker is concerned with the size of the equipment. Usually, several sizes of equipment are available from which he selects the most appropriate. This situation is common in the case of chemical equipment, where economies of scale often exist.

The second type of investment decisions applies to the situation where economies of scale cannot be obtained by increasing the output rate of production units. The increase in capacity is made by additions of standard equipment. In this case, there is only one size, so the management decision is concerned with the number of facilities, rather than the size. That is, the increase in capacity of a plant is not made by substituting larger and more efficient facilities, but by duplicating equipment of the size already existing. This situation is illustrated in a later chapter.

1.3 Objective

The objective of this study is twofold: (a) to develop a production-capacity planning model which describes the operation and expansion of a multi-product, multi-plant firm over a planning horizon, and (b) to propose a method of solution of the model in order to determine the "best" allocation of capacity increases in the system, to maximize the present value of the firm's profit, and to satisfy the forecasted demand. The "best" allocation refers to the selection of the best alternatives from a subset of possible courses of action. Thus, it cannot be claimed that the "best" allocation contains the true optimum. However, according to March and Simon [1], the decision maker is usually more interested in working with satisfactory alternatives than in working with true optimal alternatives.

There is a trade-off between these two objectives. On one hand, it is recognized that the relevance of the model increases with its complexity. On the other hand, there is obviously no reason to construct a very complex model which cannot be solved. Therefore, a compromise exists between the complexity of the model and the practicality of its solution.

The development of the model starts with the basic assumption that the decision maker has a set of feasible alternatives for expansion from which he wants to select the most suitable subset. Several factors affecting the expansion of a firm have been included in the model. Other factors may be taken into account in a later stage, or may be tested in a sensitivity analysis. It is assumed that the increases or decreases in capacity in a plant are made in discrete steps (e.g., by adding a new production line or by closing an old plant) and that the sizes of these increases or decreases are fixed and known.

The solution of the model will provide a production-expansion plan. Thus, one part of the solution indicates which plant should satisfy the demand for which zone, in a given period of time. That is, it indicates what product mix each plant in the system should produce. The other part of the solution selects those alternatives for capacity expansion that will provide the best investment planning during a planning horizon. In summary, some of the questions which will be answered, provided that there is a set of possible alternatives, the following are: How many plants should be built? What size? What should be the product mix of each plant? What is the effect of closing an existing plant?

1.4 Importance of the Problem

The amount of literature published in the past decade indicates the concern of researchers and management about ascertaining the optimal plant size. However, most of the cases reported assume the operation of a single-plant, single-product firm. Reports of studies involving the more general situation of the multi-plant, multi-product firm have appeared only recently--mainly because of the limitations of methods to solve the models. A procedure usually used to handle the multi-plant, or in general a multi-product, investment is to calculate the figure of merit for individual projects without taking into consideration their interrelationships and the desirability of scheduling them over a period of time. As stated previously, this procedure has the obvious defect of ignoring the effect each investment has on the rest of the system, thus producing a suboptimization. Therefore, a method is needed which indicates those projects which are most likely to fit the available resources of a firm and at the same time serve to maximize profit. The model should take into account the interactions between alternative projects and, in particular, the opportunity cost of using resources in one alternative rather than another. The time also should be considered because the act of building a plant is irreversible, and also because the model should anticipate future expansions in which the ideal plant size concept will be included.

The fact that many investment decisions are made on a purely subjective basis indicates that a quantitative model of the nature described above will be useful to screen the set of possible alternatives. This does not mean that the solution of the model represents the optimal solution in a real-world problem. However, the use of this model will permit the management to see the effects of some alternatives on the system. In this way, management will have more facts to aid it in making a decision.

1.5 Method of Procedure

The main developments concerned with planning for expansion of a multi-plant, multi-product firm were classified by means of a literature search. Once the literature survey was finished, the models were developed. It was convenient to begin with the simplest case, and then relax the assumptions to include more general situations. Four models were developed. Model A represents the situation for an expansion in a single period. Model B shows the case when a planning horizon divided in several periods is considered. An extension of Model B permits producing for inventory; this is Model C. Finally, Model D represents a situation when the investments change the variable cost of a plant.

It was observed that all models fit the framework of mixed-integer problems. Therefore, a survey was made to look for an efficient algorithm to solve this type of problem. It was found that the branch and bound methods have had good results in mixed problems, where there are just a few 0-1 integer variables. This is the case for the expansion planning problems. In particular, the algorithm by Rebelin was used to solve the example presented in the last chapter of this work, where a real situation was modeled and solved. The effects on the system of assigning capacity additions to different sites, the closing of existing plants, and an increase in demand in some region were studied. In a similar fashion, the management of a firm may ascertain, to some extent, the change in profit or cost resulting from different decisions made to meet the demands in the market.

1.6 General Limitations

1. The following data are assumed to be known:

   a) The forecasted demand for each product during each period of the planning horizon.

   b) The cost of a new plant and of additions.

2. A set of feasible alternatives is available for either increases or decreases in the capacity of plants. Locations for new plants are fixed, i.e., the location problem is not considered.

3. The planning horizon is divided in time periods. Within each period, all parameters (costs, activity coefficients, etc.) are assumed to be constants. But they may be different from period to period.

4. Those products with similar characteristics are grouped together into a representative product category.

5. There are no limitations in the size of the labor force.

6. The tax structure of the firm is not included. Thus, all decisions are made before taxes are calculated.

CHAPTER II

LITERATURE SURVEY

2.1 Introduction

Since the beginning of management science, there has been a keen interest in the development of models and solution algorithms for capacity expansion and production planning problems. In order to facilitate discussion, the work which has been published has been grouped into three major classes on the basis of the type of decision as characterized by the planning horizon. The classes are as follows:

a) *Short-term decisions.* Considers planning for days or weeks.

b) *Medium-term decisions.* Considers planning for weeks or months. Sometimes these decisions are called tactical decisions.

c) *Long-term decisions.* Considers planning for years. Sometimes these decisions are called strategic decisions.

Figure 2.1 shows the relationships between these types of decisions.

Short-term decisions are generally made by lower management. One of the first systematic attempts in this area was advanced by Gantt at the beginning of this century. His Gantt-charts have been used by foremen, supervisory personnel and managers since then.

To use the above or similar techniques, one first has to know what to do. That is, these techniques help management determine how to

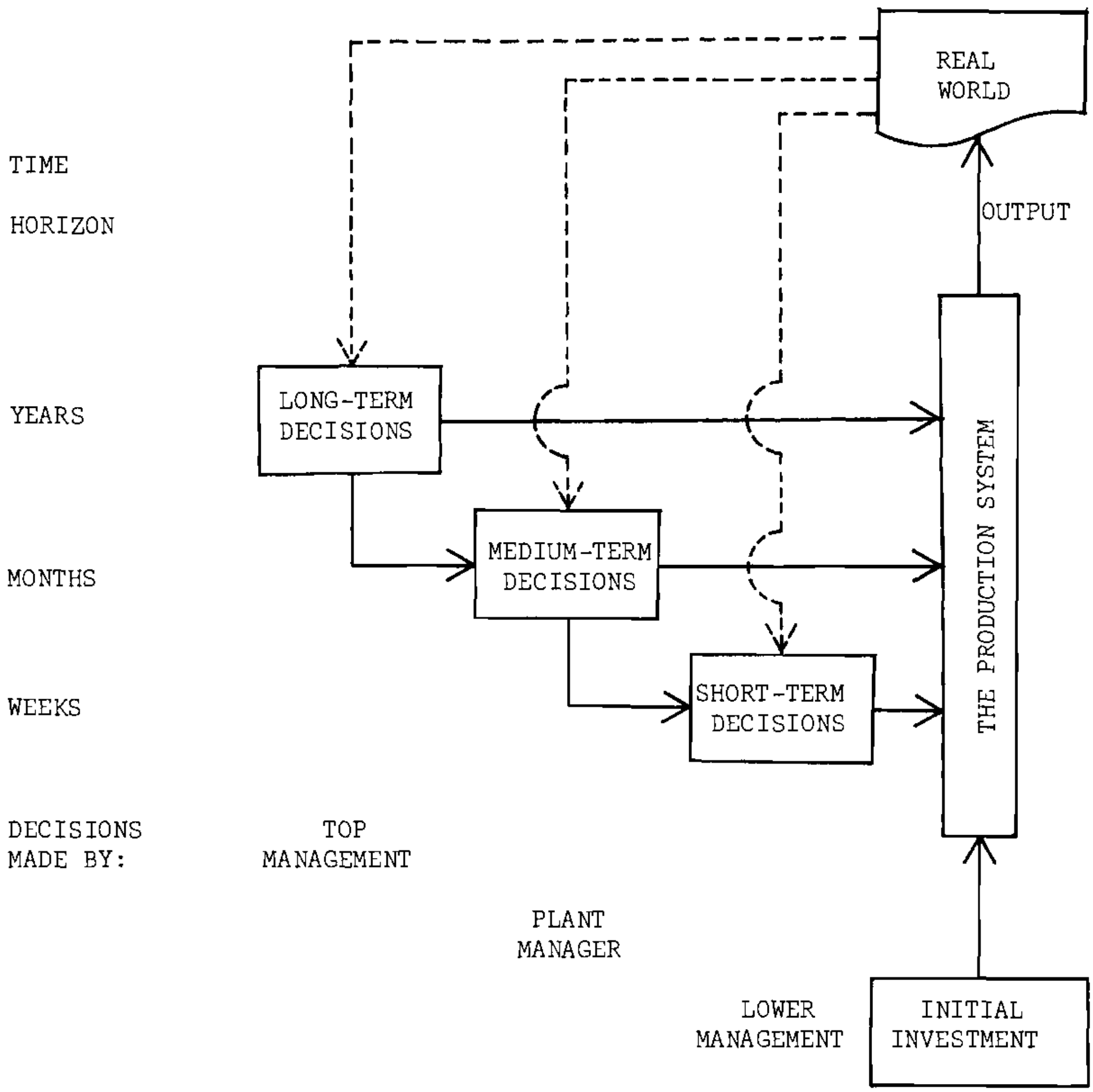

Figure 2.1 Relationships between Short, Medium, and Long-Range Decisions (Adapted from H. Chestnut, *Systems Engineering Methods*. New York: J. Wiley, 1967.)

schedule a series of jobs in the best, or at least in a satisfactory manner. They do not answer what to do, but when to do it, provided that a given sequence is satisfied.

2.2 Medium-Term Planning Decision Models

In a medium-range decision, management is more interested in combining production capacity, inventory and work force in an optimal manner so that the forecasted demand is satisfied. That is, fluctuations in orders may be followed by changes in production rate, inventory level and/or employment level. Holt, et al. [2] have reported a series of models to deal with this medium-term type of decision. The decisions to be made are tactical, in the sense that they may change the operation of the system, but the size of equipment, the number of plants, the number of lines and, in general, the kinds of equipment remain fixed. The decisions setting production rate and work force, and then inventory levels, do not involve a permanent commitment, but rather, frequent review and revision. Usually these decisions are made at regular time intervals, e.g., monthly.

The models developed by Holt, et al. [2] considered a quadratic function made up of the following costs: regular payroll, hiring and layoff of labor, overtime production and inventory. Each of these costs is weighted to override the uncertainty of the future. The sum of these weighted costs during a planning horizon is minimized, subject to a non-negative inventory constraint for each period. The authors do not consider the discount factor under the argument that a future beyond a 12-month forecast horizon has a negligible influence on the current decisions of the factory. However, if the planning horizon covers several years, the interest discounts should be considered.

Two decisions can be obtained from these models. One is the aggregate production rate and the other is the work force, both being set at the beginning of each period. Fortunately, the decision rules can be expressed in a linear equation of the type

$$P_t = K_o + \sum_{i=o}^{T-1} C_i S_{t+1} + K_1 W_{t-1} + K_2 I_{t-1}$$

and

$$W_t = L_o + \sum_{i=1}^{T-1} D_i S_{t+1} + L_1 W_{t-1} + L_2 I_{t-1}$$

where P_t is the number of units to be produced during the period t

W_p is the number of employees at the beginning of period p

I_p is the number of units of net inventory at the beginning of period p

S_p is the sales forecast for period p, and

K_i, C_i, D_i, L_i are constants.

It was noted that the constants C_i and D_i tend to zero as the period recedes from the actual date. Thus, the weight of future sales in the actual decision diminishes as the forecast extends farther into the future.

The inventory level at the beginning of a period is a link between the two decisions P_t and W_t. The production in one month affects the inventory in the next month. This in turn changes the work force in the second month, which then affects the production rate in the third month. The inventory level produces a feedback in both decisions.

The analysis of these models shows that the employment level is affected only by long-term changes in orders, while production rate responds faster to immediate changes in sales and inventory. The model was applied to a single product. For those plants producing different items, a representative product was defined and the demand, standard time, etc., for each item were expressed in terms of this representative product.

One of the first models for production planning related to the medium-range decision was presented by Bowman in 1956 [3]. The model fits in the framework of the transportation model of linear programming. Thus, the sources of production may be different plants in each period with a specified capacity, and the sinks are the aggregate demand for each period. The output of the model gives the production of each plant in each period; some of them may be to inventory.

The mathematical model is represented as follows:

Minimize cost $$C = \sum_{t=1}^{T} \sum_{q=1}^{T} \sum_{i=1}^{n} \left(C_i + C_I(q-t)\right)P_{itq}$$

Subject to the production constraints:

$$\sum_{t=1}^{q} \sum_{i=1}^{m} P_{itq} = S_q;\ q = 1,\ldots,T$$

$$\sum_{q=t}^{I} P_{itq} \leq K_i;\ i = 1,\ldots,n$$

and

$$P_{itq} \begin{cases} = 0 & \text{if } q \leq t \\ \geq 0 & \text{if } q \geq t \end{cases}$$

Where

C_i is the unit production cost of plant i

C_I is the cost per period of holding one unit in inventory

q and t represents time periods

P_{iqt} is the number of units produced by plant i during period t to be sold in period q

S_q is the number of units to be sold in period q

K_i is the capacity of the plant i (assumed to be equal for all periods).

Different approaches to the above problem have also been reported by Manne [4], Wagner and Whitin [5], and Zangwill [6]. In particular, the first two models deal with single-product production and inventory without backlogging problems. With the assumptions that the demand is known and the production and inventory cost functions are concave, the optimal production schedule is found for the next n periods, such that the cost of producing and holding inventory is minimized. Zangwill's model extends the above models to permit backlogging and to include multi-products and multi-facilities. Two algorithms are presented, one for the parallel facilities case and the other for the series facilities case.

New algorithms to obtain the optimal production plan (minimize costs) are presented by Lippman, et al. [7]. The general assumptions are that all costs are described by linear functions; the demands for the next n periods are known and are either monotonic, increasing or decreasing with time; there exist regular-time and overtime labor costs; the amount of overtime is restricted, not to exceed a fraction of the regular time utilized in any period.

3.3 Long-Term Decision Models

While the medium-term decision models give the tactical alternatives, there is a need to evaluate strategic alternatives which permit changing the structure of the system. For example, the predicted sales of a firm may indicate that an increase in output capacity of the system will be required in the next planning horizon. At this time the management of the firm may have a set of alternatives such that the system will be able to satisfy the forecasted demand.

Some alternatives for increasing the capacity of the system are:

a) additions of new plants or expansion of existing plants

b) changes in type and size of production equipment

c) changes in system configuration to improve operating efficiency.

> Examples are: (a) changes in product mix, (b) concentration of initial operations in a few centers with final production operation to be performed in satellite plants.

The selection of a satisfactory, if not optimal, alternative is a decision of far-reaching consequences. It is, in general, irreversible and its selection may interfere with the initiation of other alternatives.

The models in this section may be subclassified into two groups. The first group of models seeks to answer the strategic decision, i.e., how large a plant should be, without considering the tactical decisions. The second group of models considers both tactical and strategic decisions simultaneously. The models of the first group are suitable when the new project is independent of the existing system and are, in general, easier to apply. Some of these models are described below.

Hess and Weber [8] reported a model to find the optimal size of a one-product plant when there are economies of scale. The demand is considered probabilistic, and it is assumed that all variables can be expressed as a function of capacity. Calculus techniques are then used to determine the size of a new plant which maximizes the expected profit. A model to find the optimal plant size when the product demand growth follows the S-shaped Gomperz curve was presented by Coleman and York [9]. The demand curve is estimated with two parameters: one is the estimate of rate of growth demand, and the other the estimate of the ultimate demand level. The economies of scale are taken into account in both initial and additional investments. Thus, given the cost of a plant or an expansion as a function of its size and the parameters which define the growth of demand, the authors describe how to construct a model and a method of solution to find the initial capacity and the size of future additions so that the present value of the total cost is minimized. The uncertainty of demand is handled by running several cases with different growth parameters and then analyzing the increments in the total cost functions.

The above and similar models answer the question of the size of one plant under some given conditions. A fundamental assumption is that such a plant is independent of other plants in the system. The consideration of the interaction of several plants introduces the problem of production planning. For this reason, some research has been conducted on methods to simultaneously solve the tactical and strategic alternatives.

One of the first approaches for handling the expansion of a multi-plant, multi-product firm was advanced by Bowman [11]. A regression model was developed with the basic assumption that a relationship between unit cost of production and labor versus area served and product mix can be found and is representative of all sizes of plants. The particular system under study consisten of ten plants which supplied many products to customers located in a given region. It was desired to establish the optimal size of each plant. Some simplifications considered in the model were as follows: The production of any item was converted to units of a "product equivalent"; the total demand and product mix remained fixed; no time factor was considered. The measure of effectiveness was the cost per unit of product equivalent.

It was found that the variables with major influences on differences in the cost per unit between plants were: volume of product manufactured in each plant (V), area served by each plant (A), product mix in each plant (M), and labor (L). Since most of the cost of transformation was labor cost, the first three variables mentioned above would increase as labor costs increased. Then the mathematical model may be expressed as

$$C = L(a+b/V+cA+eM)$$

where a, b, c, e are constants and

C = annual cost of manufacturing and distribution (not including raw material) per unit of product equivalent

L = labor rates (average hourly wage payment)

V = annual volume manufactured and distributed (units of product equivalent)

A = area serviced (square miles)

M = product mix, expressed as a ratio of the average "labor content" of the plant's product to the labor content of the company's least costly product.

The underlying rationale of the analysis is that the set of plants examined is a small sample from an infinite number of similar (possible) plants. Therefore, the parameters a, b, c belong to all the systems of possible plants. In particular, they were estimated by a least squares multiple regression. Once these parameters have been established, the minimum cost for a plant can be calculated by setting the values of (L,V,A,M).

The model was used to predict the cost of the plant in operation and then compared with the actual cost. Once it was determined that the model could predict the cost per unit of product equivalent for a given set of variables L,V,A, and M, the next step was to see how the variables of the model could be manipulated to minimize the unit cost. In case of changing the volume of a plant, the labor rates and product mix would not necessarily vary, but this is not true for the area served, because the greater the volume, the greater the number of customers required to absorb the additional increase of volume. The relation between volume and area was established $K = V/A$ and called "sales density" and was assumed as a constant within each plant's territory. By substituting $A = V/K$ in the model and taking the derivative dC/dV and equating it to zero, the optimum volume is found $V = K^{1/3}\ (2b/c)^{2/3}$. The optimum value is thus a function of the sales density and the parameters b and c. The application of this formula in the existing plants resulted in the conclusion that most of them were too small (the ratio of actual size to optimum size varying from .15 to .72), and therefore for the given or fixed demand, a better number of plants would be half of the actual number, each operating at optimum volume.

A sensitivity analysis of the cost vs. volume showed that a plant that was as much as 50 per cent too small incurred an increased cost of 5 to 8 per cent, while one 50 per cent larger than the optimum size increased costs by only 1 to 2 per cent. Therefore, the analysis indicated that the plants should be larger and that if an error is made, it is better to make it on the large side.

The author pointed out that while the optimum size of the plants would not be recommended as precise answers, the model indicated that the company's plants were too small and that a system with fewer and larger plants would be a less costly one.

Another model using regression analysis was reported by Lawless and Hass [11]. With quite different objectives from the above model, the authors identified significant variables affecting the problem of size of a one-plant, one-product firm. They found the equations for the present value of cash flows incurred in the investment in new plants and their expansions during a period of time. Based upon these equations, a series of nomographs was presented which provided a means for graphical solution. It was assumed that the sales forecast of one product for a planning horizon was known and could be expressed as a compound percentage increase for a year. They later studied the situation in which the actual demand was different from that forecasted.

The problem was stated as "to choose the most economic of the alternatives which will provide the capacity needed to satisfy the actual sales requirements (which may or may not be equal to the sales forecast)." The set of possible courses of action was reduced to four, namely,

a) Build a plant to match the six-year sales forecast.

b) Build a plant to match the three-year sales forecast and add one increment of expansion during the third year to satisfy the six-year requirement, if needed.

c) Build a plant to match the two-year sales forecast and then build additional increments during the second and fourth years, if needed.

d) Build a minimum-size plant for the first-year demand and add an increment each year for five years, if needed.

Decisions b, c and d provided flexibility so that the cost of new expansion might be avoided if the demand did not develop as forecasted.

By comparing the differences among the four alternatives, those economic factors which were equal for all alternatives could be ignored. Accordingly, income from sales and the cost, which vary directly with volume, did not need to be considered. Since the timing of expansion was included, the present value method was chosen to compare the alternatives.

Ten variables which affected the present value of the cost of the alternatives were examined. They are: 1) project life, considered as 15 years; 2) taxes, 52 per cent; 3) depreciation, double declining method at 10 per cent; 4) construction time, 18 months for a new plant and 12 months for an addition; 5) projected growth rate, a range between 5 and 20 per cent; 6) actual growth rate, six comparisons vs. projected growth rate; 7) scale-up factors, ratio cost = $(\text{ratio capacity})^a$, with $.6 < a < 1$; 8) scale-up factors for subsequent additions, 9) capital cost, 10 to 20 per cent of the total cost; 10) interest rate of return, a range of 10 to 20 per cent.

For each of the four possible alternatives and each of the six differences in growth rate, a factorial design layout was formed and the present value was calculated for three values of each variable. Then a linear model was fitted to the data obtained. It was found that by transforming present values to their logarithms, the model accounted for 96 to 99 per cent of the total variations. For example, an equation to find the present value of a decision, given some growth rate, is of the type

$$PV = \text{antilog}\ (3.27 + .99g - .54k + .037m)$$

These equations were drawn in a nomograph form for easier use.

In 1958, a model for the planning of an optimal structure of the French oil industry was reported [12]. Before World War II, the French oil refineries were located along the Atlantic and Mediterranean coasts. Due to the new sources of crude oil in the Middle East and the increase in demand for oil products, it was felt that the French oil industry needed to plan for expansion. Since the need for additions in capacity was evident, the problem was where to allocate these increases. Two types of alternatives were studied. The first was to expand the coast refineries and ship the final products to the consumer centers. The other alternative was to construct new refineries inland near consumer centers. This involved the selection of refinery location and size, and also the type of distribution network which should be used to transport the crude oil. Of course, an additional alternative was to use a mixed type of decision, that is, expansion of existing coast facilities and construction of new refineries inland.

The planning horizon was established at one decade. The criterion used was to minimize the total cost of making the refined products available at consumption points. Instead of working with different periods of time during the planning horizon, the model was simplified to find the optimal structure of the oil refinery systems at half of the planning horizon. Thus, a dynamic problem was converted to a static problem. Other simplifications were made by considering only a part of the total market where, it was believed, a large increase in demand was evident. There were 30 centers in this critical zone. Also, instead of working with many final products, four broad categories of final products were defined, and only four sources of raw material were considered.

There were two types of decisions. Strategic decisions specified the location of new refineries and the pipeline to distribute the refined products or the addition in capacity in existing plants. The other type of decision was called a tactical decision. While the tactical decisions are described by continuous variables, the strategic decisions are discrete variables.

For a given strategic alternative a linear programming model was developed. Its objective function was to minimize the total cost of the system subject to: a) supplying the demand at the consumer point,

b) the existing availability of raw material, and c) the capacity of each refinery.

The method used to solve the problem was partial enumeration. A list of the most promising strategic alternatives was set up, and for each one the optimal tactical decision usable was obtained by running a standard linear programming model. The strategic alternative with minimum cost was then the optimal solution. This approach was used because at that date (1957) no efficient mathematical techniques were available to solve this mixed-integer problem.

The problem to find the best plant location from a given set of locations, under increasing returns of scale, has been studied by Manne (13) and Bergendahl (14). The method of solution seeks to answer questions of where and how much capacity should be installed by a firm in order to minimize the sum of transportation and annual investment costs. The method of solution by Manne, called SAOPMA (steepest ascent one-point move algorithm), works well for small problems, but as the number of plant sites and markets grows the method becomes very efficient.

Bergendahl proposed a recursive application of separable programming to solve the plant location problem. Multiple levels of fixed investment are introduced in each plant to find how different ratios of transportation costs and plant investments influence the solution of the problem. The relation between annual investment costs and plant size is represented by a piecewise linear function with three points. Point 0 indicates no investment, point 1 that the investment has been made but not yet utilized, and point 2 that the plant is fully utilized.

Three methods are compared with two examples. The first applies the commonly used separable programming algorithm and gives a local optimum very far from the global optimum. The second method, called the two-phase solution method, consists of solving first the transportation part of the problem, then taking its optimal solution as the initial basis of the second phase. This phase is then solved by the normal separable algorithm. The results indicate that the two-phase method gives a near-to-global optimal solution for those cases in which the fixed charges are relatively small with respect to the transportation costs. Since in many cases the investment charges take a larger part of the total cost, this method is not generally applicable.

The idea of solving the problem in several phases is extended in the third method, called marginal cost parameterization. The initial concept was advanced by Day in his work on recursive programming. In a few words, this method consists of applying the separable programming algorithm many times, with the feature that the optimal solution of one phase is the basis for a starting point in solving the next. Between successive phases, the problem is slightly modified. In the initial phase, no fixed cost is considered. That is, the plant investment costs are made variable costs over the whole possible size of the plant. In this case, the investment costs are not covered, since the total size of the plant is not fully utilized. In the successive phases, the fixed costs are covered by fewer and fewer units of output until a point in which the first unit will pay for the whole plant investment. Thus, the indivisible investment is made divisible in the early phases until it becomes indivisible again in the last phase. The results indicate that this procedure gives the optimal, or at least near optimal, solution. (The optimal solutions of the examples were found by total enumeration so that they were known previously to the application of these methods.) However, the above approaches do not solve the planning problem of a multi-product firm.

An investment model which considers multi-period planning was reported by Kendrick [15]. The model considers a firm with many plants, each one able to produce many articles. Emphasis is placed upon the idea that the planning should include space and time factors. The space factor means that the actual location of plants, the transportation costs of raw material and final products, the intershipment of intermediate products, competition and complementation of plants, etc., are considered in the model.

The time factor means that multi-period planning is selected so that the desirability of scheduling projects may be calculated. However, since the data for demand, costs, technological changes, etc., have to be forecasted, a long multi-period duration may give unrealistic results.

An outcome of this model is that management may be able to study the effects on the firm of various combinations of projects and the effects of different schedules of investments over time. The purpose of the investigation was to develop a computationally feasible investment model for analyzing groups of interdependent projects and to apply the model to the steel industry in Brazil. The problem under study was defined as follows:

> A planner responsible for investment analysis in an industrial sector sends to the plants projections of the requirements for the industry's products for the next decade. He receives from the plants feasible studies on a group of investment projects, some of which are for additions to capacity and others for the establishment of new plants. The planner must then choose from among the projects and schedule them over time in such a way as to minimize the total investment, production and transportation cost and at the same time continually fulfilling the market requirements.

The author started with the description of single-period model in which three major steel plants and their markets were included. The factors considered in the model were:

a) production requirements for each product in the market

b) the capacity of all major productive units in the system

c) the cost of production for each of the products in every plant

d) the cost of transportation

e) the cost of importing the product to the market areas

f) expected profits on exports, and

g) cost of shipment of intermediate products between plants.

The tactical decisions in this model were concentrated in three areas:

a) the intershipment of intermediate products

b) the importing of final or semifinished products, and

c) the exporting of final products.

These decisions were set because of the strong economies of scale present in the steel industries. Thus, a decision might be to import a final or semifinished product up to the point that additional capacity was economically desirable.

The author then explained the solution of a single-period model. An extension was introduced with the time factor. The time horizon was divided into several periods of time, within each of which the data were assumed constant. Two additional complexities were added when the time factor entered into the model; first, the cost and demands for each product should be forecast and, secondly, the time discounting for cost should be applied in each period of time.

The author pointed out the relationships between spatial and time investment decisions with economies of scale. Thus, in spatial planning, strong economies of scale and low transportation cost would suggest the establishment of few large plants, whereas, weak economies of scale and high transportation cost would suggest many small plants located near the customer. In time planning, strong economies of scale would imply investment in large plants separated over time.

Kendrick's multi-period model fits in the framework of mixed-integer programming. The author solved the problems using Driebeek's algorithm.

Some assumptions underlying Kendrick's model are:

a) The model is deterministic.

b) The variable cost does not change with investment decisions or with the level of production.

c) No inventory from one period to another is allowed.

d) There is no limitation of available funds for investment.

e) The price elasticity of the demand is zero.

f) There is no limitation in exportation or importation of product.

An application of the expansion model involving aluminum production has been reported by Kaiser Aluminum [16]. The aluminum process consists of two stages. The first stage produces pure aluminum oxide (alumina) from a bauxite ore. The sources of this raw material are scattered around the world. The second stage reduces the alumina to pure aluminum. Thus, the firm can be considered as a multi-plant, one-product firm. The total demand for the final product is concentrated in 38 regions. A major assumption is that the demand is known and that the facilities should be expanded sufficiently to meet this demand.

Because of the power and space restrictions on expanding existing sites, and because of the growing world demand for aluminum, the management felt that development of new reduction sites was desirable. The entry of a new site originates costs such as development of roads, power lines, etc., and new overhead costs such as plant management, accounting staff, insurance, etc. The site development cost and the present value of the fixed overhead cost are lumped in a figure which is called a site-entry cost. This cost occurs only once for each new reduction site.

Since economies of scale are present in the investment of alumina facilities, and a reduction of weight is obtained in the process, the trend is to have very few large plants, near the source of raw material. The significant question is which alumina plant should be expanded to supply the reduction plants at least cost. The authors described the past procedure used by the firm, which is valued considering each site independently by assuming 1) an expansion of the site by a given amount, 2) a "logical" alumina source, and 3) a "logical destination" of the aluminum output. Then, the cost for each possible alternative was determined and the sites were ranked according to the rate of return. That approach had the defect of ignoring the interrelationships of each expansion on the rest of the system.

Hinomoto [17] has studied the case of expansion of productive capacity in discrete steps when equipment is subject to technological improvements. The basic assumptions of this model were that technological improvement is continuous over the planning horizon and that it decreases the operating costs, that the size of the facility can be treated as a continuous variable, and that the future demand is expressed as a function of time and price. The author then applied calculus to find the necessary conditions for optimum values of the following decision variables: the sizes of facilities to be added, the timing of purchases of these facilities, the length of the planning horizon, and the volume of production at each moment. The criterion to be optimized is the present value of the profit over the planning horizon.

CHAPTER III

MODELS FOR EXPANSION PLANNING

1. Introduction

As stated in Chapter I, the problem set forth in this study is to obtain the optimum expansion plan for an operating production system over a given planning horizon and with a given set of exogenous conditions. The operating system is understood to consist of sources of raw material (mills) and sources of semifinished and final products (plants), along with a set of sinks (markets) and the relationships among them. Some of the exogenous conditions that directly affect the system are the increase (or decrease) in the demand for each product; the production and delivery costs for each product manufactured in each plant; technology changes in the manufacturing processes; and the cost of the capital.

In order to simplify the solution of the problem, the planning horizon is divided into a finite number of periods. It is assumed that within a period the exogenous and internal conditions remain constant. In some production systems, it will be necessary to include many periods in order to have a realistic representation of the real world; in others, a few periods are sufficient to have a good approximation of the real situation. The more periods considered, the better is the approximation of the real problem, but the more difficult it is to solve. There is a trade-off between number of periods and time required to amass the data and solve the problem.

In order to meet the changes in demand, it is necessary to supply some alternatives by means of which the production system will be able to satisfy these requirements in an "optimal" manner. Usually, these alternatives consist of projects that have passed several tests and are the most promising as evaluated by management or the people who make the decisions. Thus, the word "optimal" is used in a restricted sense, because the optimization is made on the set of proposed alternatives, and there is no assurance that the true optimal will be an element of such set.

This procedure of selecting alternatives agrees with the hypothesis advanced by March and Simon [1], who state that the decision process of an individual is more concerned with the discovery and selection of *satisfactory* alternatives than with the discovery and selection of *optimal* alternatives.

The alternatives considered in the models to be described later, can be classified in the following groups:

a) *New Investments.* Either new plants or additions to the existing ones may be added to the system. Usually these alternatives require a large investment of money, and are long-range decisions.

b) *Intershipments between Plants.* The excess in capacity in a given department or in a "surplus" plant may be used to produce a semifinished material needed by a "deficit" plant in the system.

c) *Rented Equipment.* This alternative differs from new investments in that this project is more flexible and may require less investment in equipment. In some cases it may be considered as a short-range decision.

d) *Inventory.* Goods produced in one period may be carried over to the next period. In industries that work on request, the inventory buildup may not be economically feasible because of the high risk involved in keeping a product which may not be requested in other periods.

When a decrease in demand in one or several products is forecast, the following additional alternatives are considered:

e) *Close a Facility.* The facility to be closed may be a plant, a department, a line, etc.

f) *Reallocate Facilities.* Some equipment may be utilized in other types of production.

Four models are presented in this chapter. Model A is the simplest because the time factor is not included. However, the development of this model will be used to define most of the variables and parameters of the system. Model B describes the situation in which the planning horizon is divided into several time periods. In each period a set of investment projects is proposed, but no inventory is carried from one period to another. This restriction is relaxed in Model C, which includes several time periods and also provides for inventory in order to satisfy some demands in subsequent periods. Finally, Model D considers the case in which the unit variable cost depends upon the output of a plant and upon capacity changes resulting from investment decisions in a given time period. Model D is a variant of Model C. A more general model would be a combination of Model D and Model C.

The models are called as follows:

A. Single-Period Model

B. Multi-Period Model

C. Multi-Period with Inventory Model

D. Multi-Period with Changes in the Unit Variable Cost Model

The relationships between the models are described below:

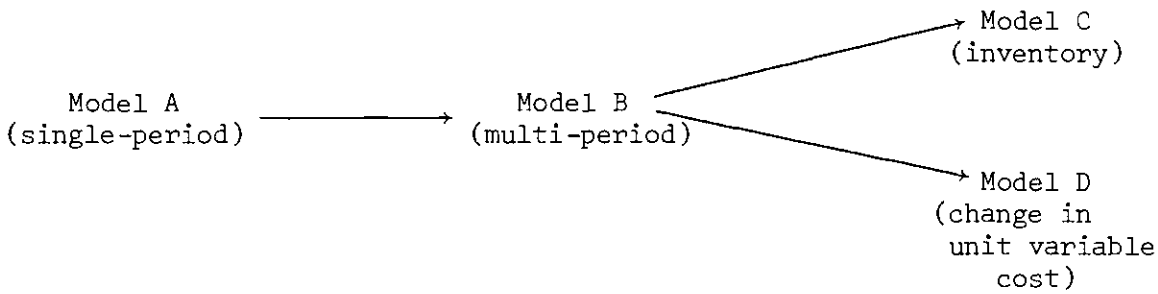

Figure 3.1 Relationships between the Four Models

3.2 Subsystems, Variables and Parameters

Before presenting the models, it is convenient to define the system, its variables and parameters. All models consider the production-consumer system of at least the following elements:

a) a set of sources of raw materials,

b) a set of existing plants (within each of which there exists a given set of departments or processes),

c) a set of markets or centers of consumption,

d) a number of different products,

e) a given demand for each product in each market for each period,

f) a set of alternatives for expansion from which a subset is to be chosen.

The total system, including vendors, plants, satellite plants and consumers, may be represented as in Figure 3.2.

Consider now an isolated plant. It receives raw materials from vendors and semifinished products from other plants. Its production is shipped to other plants and/or consumer markets. Figure 3.3 represents the inputs and output of a plant p. The amount of raw material f shipped from source i to plant p is indicated by the variable W_{fip}. The amount of semifinished product m' received from plant p' is shown by variable $U_{m'p'p}$. The outputs of plant p are S_{mpr} and $U_{m''pp''}$. The former indicates the amount of finished product m shipped to market r. The latter represents the amount of semifinished product m" shipped to the other plant p".

Thus, a plant may receive input in the form of raw materials or semifinished products, and send its output to consumer centers and other plants.

3.3 Notation

Table 3.1 shows the symbols used to designate the subsystems and elements of the system. These symbols appear as subscripts in the parameters and variables shown in Tables 3.2 and 3.3, respectively.

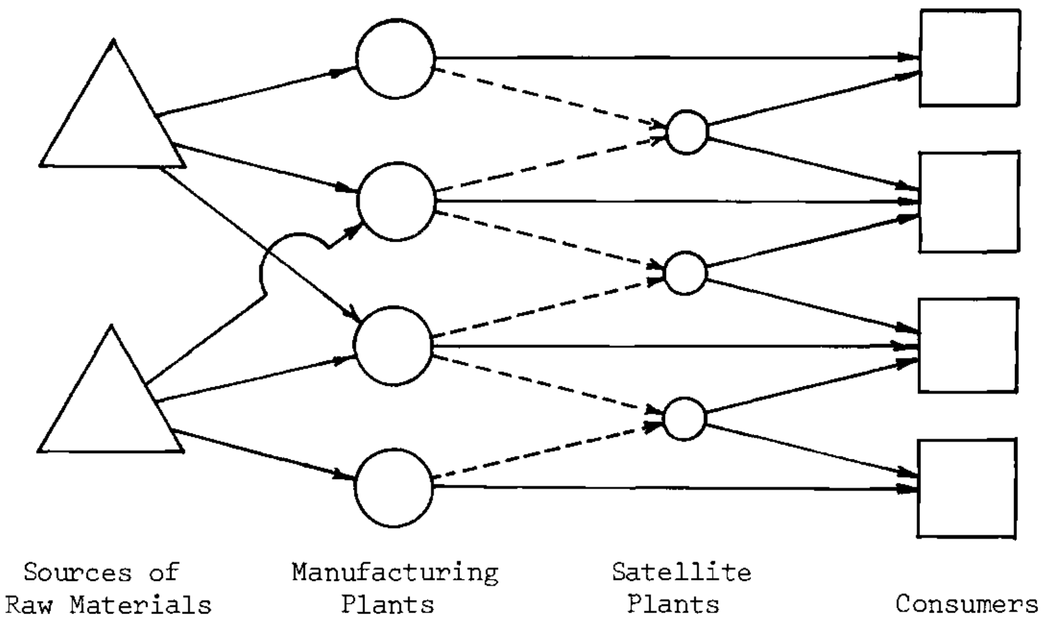

Figure 3-2. The Production-Consumer System

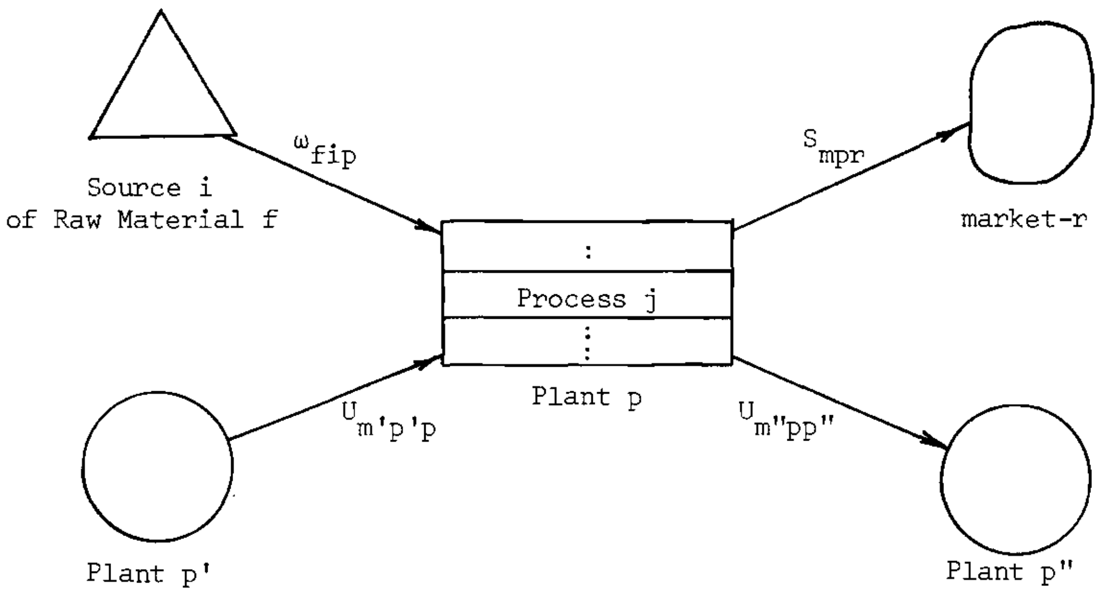

Figure 3-3. An Isolated Plant in the Production-Consumer System

Table 3.1 Subsystems and Elements of the Production-Consumer System

Sources of input:	$i = 1,2,\ldots,k$ (raw material)
	$p = k+1,\ldots,I$ (plants)
Departments or process within a plant p, p(j):	$j = 1,2,\ldots,J_p$
Raw materials:	$f = 1,2,\ldots,F$
Intermediate products:	$m = 1,2,\ldots,\mu$
Final products:	$m = \mu+1,\ldots,M$
Consumer centers:	$r = 1,2,\ldots,R$ (markets)
Projects for expansion at plant p:	(Alternatives) $v_p = 1,2,\ldots,v_p'$
Projects for renting equipment at plant p:	$v_p = v_p' + 1,\ldots,V_p$

Table 3.2 Parameters of the Production-Consumer System (One Period)

$b_{p(j)}$	= Capacity of department or process j in plant p
$\beta_{p(j)v_p}$	= Capacity created or reduced in process j at plant p by project v_p
d_{mr}	= Requirements of product m in market (consumer center) r
$a_{mp(j)}$	= Capacity required per unit m, in plant p, process j
e_{fm}	= Units of raw material f used to produce a unit of product m
c_{mpr}	= Cost of producing a unit or product m, in plant p and shipping to region r (cost of product f.o.b. in market r; not real cost if shipments of intermediate product are involved)
h_{mpn}	= Additional cost of using intermediate product m, in plant m, which was produced at plant p
q_{fin}	= Cost of raw material f shipped from source i to plant n
π_{mr}	= Price of product m, in market r
g_{v_p}	= Cost of investment of project v_p, in plant p
D_{v_p}	= Capital recovery factor for investment project v_p, i.e. $\rho(1+\rho)^L/\{(1+\rho)^L-1\}$, where ρ is the interest rate of the firm and L is the expected life of project.

Table 3.3 Variables of the Production-Consumer System

S_{mpr}	= Amount of units m produced at plant p and shipped to market r
U_{mpn}	= Amount of intermediate units m produced at plant p and shipped to plant n
X_{v_p}	= Investment or renting decision variable for project v_p at plant p (this variable can only take value 0, 1)
W_{fip}	= Amount of raw material f shipped from source i to plant p

A first step in the expansion planning of the production-consumer system is to define 1) the sources of input, either raw materials or semifinished products; 2) the sources of output, i.e., the number of plants and departments within each plant; 3) the number of products, raw materials, semifinished and finished products; and 4) the proposed increases or decreases in the capacity of the system. Any parameter or variable to be considered must specify at least one of these elements.

In general, capital letters are used to indicate variables and limits in the subscripts. Small letters represent parameters and subscripts.

3.4 General Assumptions Underlying All Models

The general assumptions applying to all models are given below. Those applicable to a particular model are presented just before its development.

1. The models are deterministic. That is, the parameters and variables within a given period of time are invariant. But they may change from one period to another.

2. The terms associated with continuous variables are linear. That is, if the production cost of a unit m_1 is c_1 and of a unit m_2 is c_2, then the total cost of producing k_1 units of m_1 and k_2 units of m_2 is $k_1c_1 + k_2c_2$. Also, the total cost of the system is represented as the sum of the total costs of its plants, i.e., the total cost function is a separable function.

3. The capacity of a process within a plant is defined as the ability to satisfy a given requirement. The potential capacity of a process is the sum of the manufacturing capacity plus inventory on hand, plus additional equipment. For example, if the original capacity of process A is 100 units/year and there is an initial inventory of 20 units, the potential capacity of the process is 120 units.

4. The additional capacity added in a process by new investment remains constant during the minimum of the planning horizon and the life of the equipment. The life of rented equipment is equal to one period. If a plant or a process is closed down, the capacity of the system will be diminished by the corresponding quantity for the remaining periods.

5. All decisions are made at the beginning of a period.

6. The cost of a given alternative is charged as a discounted uniform annual cost over the expected life of the equipment.

The objective function in each model is to maximize the profit of the system, and the variables are subject to six general groups of constraints:

Market Group

The demand for each product in each region must be satisfied. If the original capacity of the system is not sufficient, additional capacity in the form of new investments, rented equipment, inventory, etc., will be introduced in the model.

Raw Materials Group

There is a limited amount of raw material available at a particular source. However, total material from all sources is sufficient for production requirement.

Availability of Funds Group

Funds expended for capital expansion in any period must not exceed the money available for investment in this period.

Capacity Group

The original capacity plus the additions or reductions in capacity made as a result of capital investment decisions must be sufficient to meet the forecast demand.

Integer Group

The decisions to be made concerning new investments and rented equipment are designated by zero-one variables.

Inventory Group

The amount of material used from inventory must not be greater than the inventory available at the beginning of a period.

3.5 Model A: Single Period

The single-period model represents a simple situation in which the decisions are made for one single period.

Assumptions

1. Single period.

2. The increase in capacity in this period may be made by new investments or rental of equipment.

Parameters

Same as Table 3.2.

Variables

Same as Table 3.3.

Objective

The objective function is a profit equation as follows:

$$Z_A = \sum_{r=1}^{R} \sum_{m=\mu+1}^{M} \sum_{p=k+1}^{I} \pi_{mr} S_{mpr}$$

(total revenue obtained by total sales)

$$- \sum_{r=1}^{R} \sum_{p=k+1}^{I} \sum_{m=\mu+1}^{M} c_{mpr} S_{mpr}$$

(total cost of production of final products)

$$- \sum_{n=k+1}^{I} \sum_{\substack{p=k+1 \\ p\neq n}}^{I} \sum_{m=1}^{\mu} h_{mpn} U_{mpn}$$

(total additional cost for shipping intermediate products)

$$- \sum_{n=k+1}^{I} \sum_{i=1}^{k} \sum_{f=1}^{F} q_{fin} W_{fin}$$

(total cost of raw materials)

$$- \sum_{i=k+1}^{I} \sum_{v_p=1}^{V_p} D_{v_p} g_{v_p} x_{v_p}$$

(total capital cost for investments)

The variable S_{mpr} in the first term is the number of units m sold at market r from plant p. The sum over all plants times the price gives the total sales (in dollars) of product m in market r. The double summation over final products and over markets gives the total revenue.

The second term ("total cost of production of final products") includes only the cost added during manufacturing; it does not include the cost of raw materials.

The additional cost for intershipment and intermediate product is the difference in unit cost of the receiving plant, minus the unit cost f.o.b. the supplier plant, plus transportation costs. Summation over products and a double summation over plants gives the third term.

The cost of all raw materials used in the system is given in the fourth term. Finally, the fifth term represents the portion of the investment g_{v_p} corresponding to this period. The sum over all projects gives the total cost of investments.

Note that once a project v_p is selected, the parameters D_{v_p} and g_{v_p} are completely defined. The parameter D_{v_p} is the recovery factor and depends only on the rate of interest and the life of the project v_p. The total cost of the project g_{v_p} includes not only the cost of equipment, construction, etc., but also the cost of planning.

It should be stressed that only the portion $D_{v_p} g_{v_p}$ of the total cost of a project is charged in a period. That is, the system will pay only for the time it uses the investment. If v_p is an alternative corresponding to rental of a piece of equipment, the recovery factor D_{v_p} is 1, and g_{v_p} is the rental cost per period. If the renting contract includes fixed ($/period) and variable costs ($/hour) then g_{v_p} corresponds to the fixed cost. Then the additional term $g'_{v_p} H_{v_p}$ must be added to the objective function where g'_{v_p} is the variable cost and H_{v_p} the hours worked in a period by the rented equipment v_p. Also a new constraint should be included to guarantee that the equipment may be used only if the equipment is rented. That restriction is:

$$H_{v_p} \leq \beta_{p(j)v_p} X_{v_p}$$

where $\beta_{p(j)v_p}$ is the total capacity added by project v_p (i.e., hours/period) and is an upper bound on the value of H_{v_p}.

Note that $X_{v_p} = 0$ if, and only if, $H_{v_p} = 0$. The "if" part is obvious, since no utilization implies that the equipment should not be rented. The "only if" part says that if no equipment is available ($X_{v_p} = 0$), it cannot be used. In either case, no costs are incurred.

Now $H_{v_p} > 0$ if and only if $X_{v_p} = 1$, the sufficiency follows from the restriction. (Of course, H_{v_p} cannot be greater than the upper bound $\beta_{p(j)}X_p$). The necessity part assumes $X_{v_p} = 1$. If $H_{v_p} = 0$, then $X_{v_p} = 0$ as shown above, which is a contradiction; therefore, $H_{v_p} > 0$.

Constraints

This model contains four groups of constraints, namely, capacity, markets, raw materials, and integer constraints. The non-negativity constraint is assumed to be implicit in all models.

Capacity Constraints

The general constraint for a process j at plant i is given by the inequality below:

$$\underbrace{\sum_{m=\mu+1}^{M} \sum_{r=1}^{R} a_{mp(j)} S_{mpr}}_{(f)} + \underbrace{\sum_{m=1}^{\mu} \sum_{\substack{n=k+1 \\ n \neq p}}^{I} a_{mi(j)} U_{mpn}}_{(g)}$$

$$\underbrace{\sum_{m=1}^{\mu} \sum_{\substack{n=k+1 \\ n \neq p}}^{I} a_{mp(j)} U_{mnp}}_{(h)} - \underbrace{\sum_{v_p=1}^{V_p} \beta_{p(j)v_p} X_{v_p}}_{(i)} \leq b_{p(j)}$$

For all p, j.

p = k+1,...,I (plants)

j = 1,2,...,J_p (departments).

There are (I × J_p) equations.

This inequality states that the capacity utilized to produce intermediate and final products in process j, at plant i must not exceed the capacity available. In order to relax the bottlenecks of the process, two decisions can be made: 1) receive intermediate products from other plants, and 2) increase the capacity by a new investment. These relaxations in the constraints are indicated by the negative terms in the left side of the above inequality.

The term (f) represents the capacity used to produce the final products shipped to markets $r = 1,\ldots,R$. The capacity utilized to produce intermediate products is indicated in term (g). One way to ease the capacity restrictions of process j, at plant i, is by receiving some amount of intermediate products from other plants which are the same as those produced in this process. Such inter-shipments are represented by the term (h). Finally, the addition of capacity obtained by the construction and installation of a new project at department or process j at plant i is indicated by the term (i). The algebraic sum of these four terms must be less than or equal to the available capacity of the process $b_{p(j)}$.

<u>Market Constraints</u>

The sum of production over all plants for each product and each market must be enough to satisfy the demand of this product in a market. Thus the demand of product m at market r is satisfied by

$$\sum_{p=k+1}^{I} S_{mpr} = d_{mr} \qquad \text{For all}$$

$$m = \mu+1,\ldots,M \quad \text{(final products)}$$

$$r = 1,\ldots,R \quad \text{(markets)}$$

There are $(M \times R)$ equations.

Raw Material Constraints

The total of raw material f utilized at plant m to produce intermediate and final products is equal to the total received from all sources of raw material, that is,

$$\underbrace{\sum_{r=1}^{R} \sum_{m=\mu+1}^{M} e_{fm} S_{mir}}_{(j)} + \underbrace{\sum_{p=k+1}^{I} \sum_{m=1}^{\mu} e_{fm} U_{mip}}_{(k)} = \underbrace{\sum_{L=1}^{k} W_{fLi}}_{(m)}$$

For all

$f = 1,\ldots,F$ (raw material)

$i = 1,\ldots,k$ (source of material).

There are $(F \times k)$ equations in this group.

The term (j) represents the raw material f consumed in producing final products and the term (k) that consumed in making intermediate products, at plant p. The term (m) gives the total raw material f received by plant p, in this period. A total of $(F \times k)$ equations are in this group.

Integer Constraints

The variables which represent the decision of executing or not executing a given project can take only values 0 or 1.

$$X_{v_p} = 0,1; \quad \text{for } p = k+1,\ldots,I; \quad v_p = 1,\ldots,V_p$$

There are $(I \times V_p)$ equations.

Non-Negativity Constraints

$$S_{mpr}, U_{mpn}, W_{fin} \geq 0; \quad \text{for all } f,i,p,r,m,n.$$

3.6 Model B: Multi-Period

Assumptions

1. The planning horizon is divided into several periods, not necessarily equal. The interest rate on capital remains the same for all periods.

2. The increase in capacity in each period may be made by new investments and/or rented equipment.

3. In each period there is a limited amount of money available for investments.

4. The cost of production in a given period is not affected by decisions made in that period or prior periods. This cost, however, may change from period to period; for example, forecasted increases in cost of raw material, labor, etc., may be included here. However, if a new project would change the operating cost, then this model does not apply.

5. No inventory can be carried from one period to another.

Parameters

Same as Table 3.2 with the additional superscript "t" to indicate time period (t=1,...,T). For example, $b^t_{p(j)}$ is the capacity of department j, plant p, during time period t. Other parameters are:

$\alpha^t_{v_p}$ = outlay of money required at time t by investment (or rental) project v_p

γ^t = total money available to spend in investment and rental projects

δ^t = the present worth factor. If the interest rate is ρ, then $\delta^t = 1/(1+\rho)^t$

η_{v_p} = present value of project v_p; equal to $\sum_{t=1}^{L} \delta^t D_{v_p} g_{v_p}$, where L is the minimum of the life of the equipment and the planning horizon, D_{v_p} is the capital recovery factor, and g_{v_p} is the cost of project v_p.

Variables

Same as Table 3.3 with the additional superscript "t" to indicate time. For example, S^t_{mpr} represents the units of product m, produced at plant p and shipped to market r during time period t.

Because of the changing value of money over time, the cost and profits of several periods cannot be added directly. It is necessary to calculate their equivalent value at a given time. In this thesis the Present Value criterion is used. That is, all costs and profits will be converted to their equivalent values at the beginning of the first period.

Since the constraints are given for each period, the discount factor is not applied to the constraints.

Objective

Similarly to the Single-Period Model, the Multi-Period Model maximizes the profit of the system. The new objective function, which includes time, is:

$$Z_B = \sum_{t=1}^{T} D_t \{ \sum_{m=\mu+1}^{M} \sum_{p=k+1}^{I} \sum_{r=1}^{R} \pi_{mr}^{t} S_{mpr}^{t}$$ ...Total revenue from sales during the horizon planning period.

$$- \sum_{r=1}^{R} \sum_{p=k+1}^{I} \sum_{m=\mu+1}^{M} c_{mpr}^{t} S_{mpr}^{t}$$ ...Total cost of final products f.o.b. each market. It does not include raw materials nor capital cost for new projects.

$$- \sum_{n=k+1}^{I} \sum_{\substack{p=k+1 \\ p\neq n}}^{I} \sum_{m=1}^{\mu} h_{mpn}^{t} U_{mpn}^{t}$$ ...Additional cost for shipping intermediate products from plant to plant.

$$- \sum_{i=1}^{k} \sum_{f=1}^{F} \sum_{p=k+1}^{I} q_{fip}^{t} W_{fip}^{t} \}$$ ...total cost of raw materials consumed in all plants.

$$- \sum_{t=1}^{T} \sum_{p=k+1}^{I} \sum_{V_p=1}^{V_p^t} n_{v_p} X_{v_p}^{t}$$ ...total cost of all additional investment made in the horizon period.

Constraints

Capacity. The capacity restriction of department j at plant p during the period t is:

$$\sum_{m=\mu+1}^{M} \sum_{r=1}^{R} a_{mp(j)} S_{mpr}^{t} + \sum_{m=1}^{\mu} \sum_{\substack{n=k+1 \\ n\neq p}}^{I} a_{mp(j)} \mu_{mpn}^{t}$$

Capacity used at department j, plant p, during period t to produce final product

Idem, to produce intermediate products.

$$\sum_{m=1}^{\mu} \sum_{\substack{n=k+1 \\ n\neq p}}^{I} a_{mp(j)} U^{t}_{mp} \quad - \quad \sum_{\tau=1}^{t} \sum_{v_p=1}^{v_p} \beta^{\tau}_{p(j)v_p} \quad X^{\tau}_{v_p} \le b^{t}_{p(j)}$$

Capacity "received" from other plants through intermediate products	Increased capacity due to investments up to time t	Capacity available in department j, plant p at time t.

For all p, j, t.

Market

$$\sum_{p=k+1}^{I} S^{t}_{mpr} \qquad = \qquad d^{t}_{mr}$$

Total units of product m received at market r during period t.	Demand for product m at market r during period t.

For all m, r, t.

Raw Material

$$\sum_{i=1}^{k} \sum_{m=\mu+1}^{M} e_{fm} S^{t}_{mpi} \qquad + \qquad \sum_{\ell=k+1}^{I} \sum_{m=1}^{\mu} e_{fm} U^{t}_{mp\ell}$$

Raw material f consumed at plant p at time t to produce final products.	Idem, to produce intermediate products.

$$= \sum_{i=1}^{k} W^{t}_{fip} = W^{t}_{fp}$$

Total raw material f available at plant p, at time t.

For all f, p, t.

Availability of Money

$$\sum_{p=k+1}^{I} \sum_{v_p=1}^{V_p} \alpha_{v_p}^t X_{v_p}^t \qquad \leq \qquad \gamma^t$$

Total outley of money required by investment and rental projects at period t.	Available money for investments at period t.

For all t.

If different monetary limits exist for investment and for renting equipment, then two constraints are needed, one for the investments and one for the rented equipment.

Non-negativity

$$W_{fip}^t,\ S_{mpr}^t,\ U_{mpn}^t \geq 0 \text{ for all } p,r,m,n,t.$$

Integer

$$X_{v_p}^t = 0,1 \qquad \text{for: } \begin{array}{l} p=k+1,\ldots,I \\ v_p = 1,\ldots,V_p \\ t = 1,\ldots,T. \end{array}$$

The matrix layout of this model is shown in Figure 3.4. Note that the only equation common to all periods is the objective function.

The letter inside a block corresponds to a vector of variables or a vector or matrix of parameters defined previously. For simplicity, the only subscript used is the time period t. Thus,

S_t is a vector of continuous variables; each element gives the amount of a final product which each plant produces and sends to a given market.

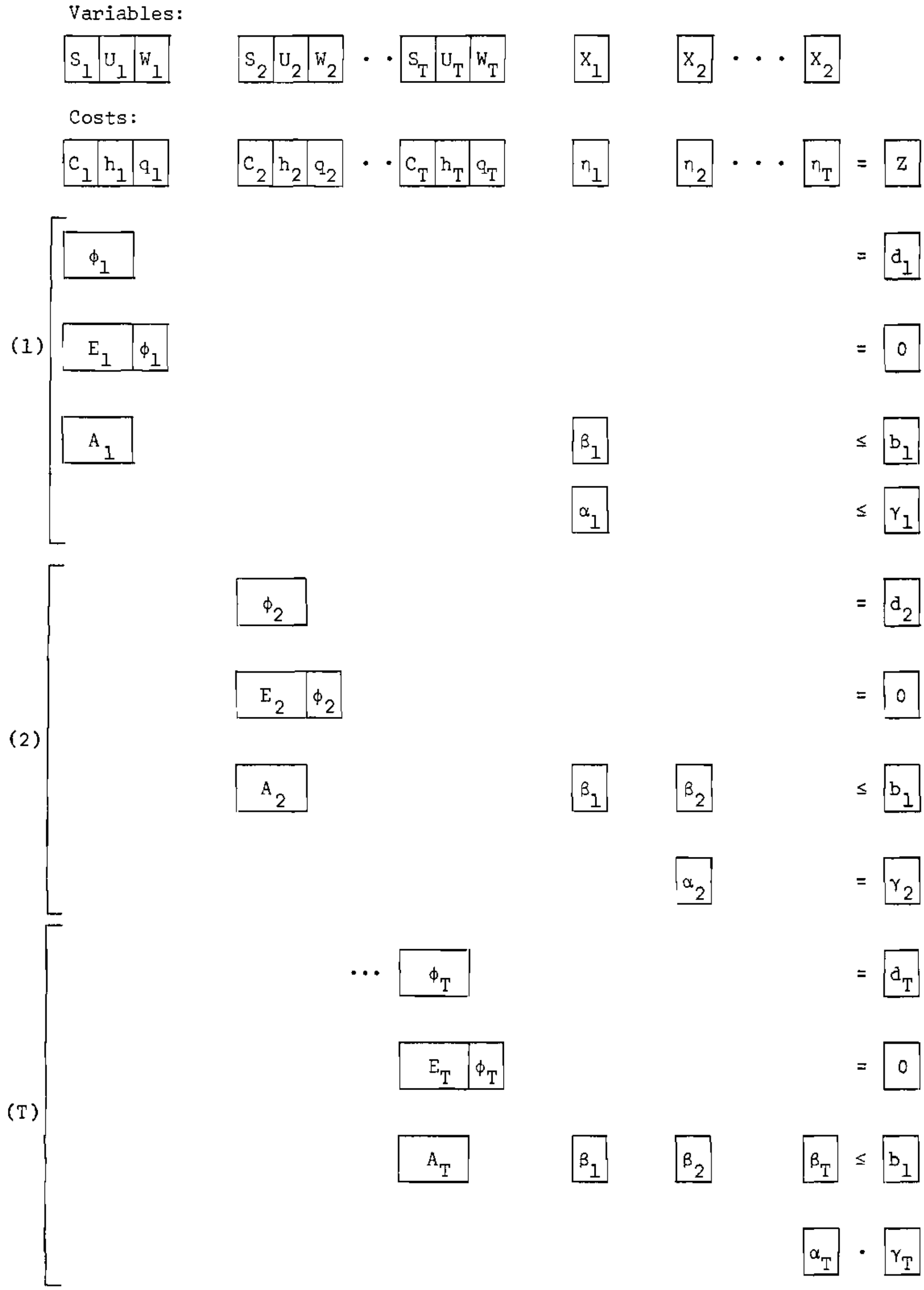

Figure 3.4 Matrix Layout of Model B

W_t is a vector of continuous variables; each element gives the amount of a raw material shipped from a source to a plant.

U_t is a vector of continuous variables; each element gives the amount of an intermediate product shipped between plants.

X_t is a vector of 0-1 variables corresponding to the investment decisions.

C_t is a vector of costs; each element gives the cost of producing the amount in S.

h_t is the cost vector for intershipment between plants.

q_t is a cost vector for raw materials.

ϕ_t is a matrix of zeros and ones; each row indicates all possible plants that can be used to satisfy the demand of a final product in a zone, or of an intermediate product in a plant.

E_t is a matrix of coefficients; each element gives the amount of raw material used by variables S_t and V_t.

ϕ_t is a matrix of zeroes and -1, whose elements give the possible sources of raw materials.

A_t is a matrix of coefficients; each element gives the amount of capacity consumed by a variable S_t or V_t.

β_t is a matrix of coefficients relating the increases or decreases in capacity when an alternative X_p is in the solution. An addition in capacity corresponds to a negative element; a decrease, a positive element.

α_t is a row vector with elements representing the outlays of money required by the projects.

b_t is a vector of original capacities of each process in each plant.

γ_t is the limit of capital available in period t.

d_t is the vector of demand.

The integer and non-negative constraints are not included in Figure 3.4. When a project v_p is paid in a stream of payments

$g_1, g_2, \ldots, g_T$, then its present value is η_{v_p}. If the project is paid during the planning stage, then g_t represents the portion of the initial capital outlay corresponding to period t, by applying the recovery factor.

3.7 Model C: Multi-Period with Inventory

Assumptions

1-4. Assumptions 1 to 4 from Model B (Section 3.4) hold.

5. Some inventory may be carried from one period to another. The input to and output from inventory is considered as a constant rate within a period of time.

Parameters

Same as Model B, plus the following:

N^1_{mp} = units of product m in inventory at plant p at the beginning of period one

N^{T+1}_{mp} = desired inventory of product m at plant p at the end of the planning horizon

σ^t_m = cost of holding one unit m in inventory during time period t

c^t_{mp} = cost of producing a unit m at plant p, process j, during time period t. Note that $c^t_{mp(j)}$ is not equal to c^t_{mpr} because the latter includes the transportation cost to deliver the product in region r.

$\hat{c}^t_{mpr}$ = cost of transporting one unit m from plant p to market τ during the period of time t. Note that without inventory $c^t_{mpr} = \hat{c}^t_{mpr} + (\sum_j c^t_{mp(j)} \underline{J}_p)$, where $\underline{J}_p$ is the number of process j at plant p, that produces the product m.

Variables

Same as Model B, plus the following:

N^t_{mp} = units of product m in plant p at the beginning of period t; equal to the inventory at the end of period t-1

$\hat{S}^t_{mpr}$ = units of product m, sent from inventory at plant p to market r during time period t

$Y^t_{mp(j)}$ = units at product m, manufactured at plant p, process j, during time period t, and sent to inventory.

Objective Function

The objective function for Model C contains the same terms as those in Model B, i.e., Z_B, plus three additional terms corresponding to the cost of producing for inventory, the cost of carrying inventory, and the cost of shipping units from inventory.

In Models A and B, the cost of production and shipping were concentrated in one figure, c^t_{mpr}. It was implicitly assumed that production and transportation occurred in the same period of time.

In Model C, this is not the case. Units may be produced in the first period, kept in inventory during the next two periods, and sold in the fourth period. Therefore, here the cost, c^t_{mpr}, is broken down into its components, which are added to the total cost of the system. When a unit is produced for inventory, two costs are incurred in the system--the cost of production and the cost of carrying inventory. For this reason, the cost associated with variable $\hat{S}^t_{mpr}$ is only the cost of transportation $\hat{c}^t_{mpr}$.

The total cost of producing to inventory during a time period t is:

$$CPI^t = \sum_{m=1}^{M} \sum_{p=k+1}^{I} \sum_{j=1}^{J_p} c^t_{mp(j)} Y^t_{mp(j)}$$

The total cost of shipping units from inventory during a period of time t is:

$$CSHP^t = \sum_{m=1}^{M} \sum_{p=k+1}^{I} \sum_{r=1}^{R} \hat{c}^t_{mpr} \hat{S}^t_{mpr}$$

Since the input and output inventory rates are assumed constant, the cost of carrying inventory may be set proportional to the average inventory in a given period of time.

The total cost of carrying inventory in one period of time t is:

$$CINV^t = (0.5) \sum_{m=1}^{M} \sum_{p=k+1}^{I} \sigma^t_m (N^t_{mp} + N^{t+1}_{mp})$$

The variables N^t_{mp} generate a new set of inventory constraints to be discussed below.

Thus, the objective function of Model C is:

$$Z_C = Z_B - \sum_t D_t (CPI^t + CSHP^t + CINV^t),$$

where Z_B is the objective function of Model B; D_t is the present worth factor; CPI^t is the total cost per period t for producing for inventory; $CSHP^t$ is the total cost per period t for transporting units from inventory.

Constraints

As stated above, a new set of constraints should be included in Model C. These constraints are described below.

Inventory

$N^t_{mp} \geq 0$; no shortages are allowed.

$$N^{t+1}_{mp} = N^t_{mp} + \sum_{j=1}^{J_p} Y^t_{mp(j)} - \sum_{r=1}^{R} \hat{S}^t_{mpr}$$

N^{t+1}_{mp}: Inventory of product m, plant p, at the beginning of period t + 1

N^t_{mp}: Idem, at the beginning of period t

$\sum_{j=1}^{J_p} Y^t_{mp(j)}$: Units of product m produced to inventory by plant p during time t

$\sum_{r=1}^{R} \hat{S}^t_{mpr}$: Units of production consumed from plant p during time t

For: $t = 1,\ldots,T$

$m = 1,\ldots,M$

$p = k+1,\ldots,I.$

Note that both the initial and final inventory levels of each product m at plant p (i.e., N^1_{mp} and N^{T+1}_{mp}, respectively) should be given. If there exist any limitations on the level of inventory, for example, because of space, capital investment, etc., then additional new constraints, such as

$$N^t_{mp} \leq UB^t_{mp} \quad \text{or} \quad \sum_m N^t_{mp} \leq UB^t_p$$

should be included, where UB^t_{mp} is an upper bound on units of product m to be held in inventory during period t, and UB^t_p is an upper bound on inventory at plant p, period t.

Capacity

The left-hand side of the capacity restriction in Model B, called $LCAP_B$, is modified because of the use of the facilities to produce for inventory; this restriction becomes

$$LCAP_B + \sum_{m=1}^{M} \gamma^t_{mp(j)} \le b^t_{p(j)}$$

Market

$$\sum_{p=k+1}^{I} (S^t_{mpr} + \hat{S}^t_{mpt}) \ge d^t_{mr}$$

Raw Material

Same as Model B.

Availability of Capital

Same as Model B.

Non-negativity

Same as Model B, plus

$$\gamma^t_{mp},\ N^t_{mp},\ \hat{S}^t_{mpr} \ge 0$$

Integer Constraints

Same as Model B.

3.8 The Effect of Increasing or Decreasing Variable Cost with Investments

In previous models, it was assumed that the variable cost of production was independent of the investments chosen. That is, no matter how large a plant or a process may be or how many investments are made, the variable cost per unit remains constant. Thus, if in a given plant, a decision is made to increase capacity, then the cost of the capacity increase in a given period is the part of the cost of investment corresponding to that period. The variable cost, however, remains the same. Figure 3.5 illustrates this for one plant in the system.

The slope of each line segment represents the variable cost (in this case, all are the same), and the intersection with the vertical axis gives the present value of the investment cost or rental cost of a new piece of equipment.

The point U_1 is the upper bound under the present investment (INV_1). An output below U_1 affects only variable cost, since the fixed costs of the equipment and facilities already in operation are considered as sunk costs. That is, no matter what the level of operation, these costs will be on the system. The total cost (TC) for an output (S) on this range is

$$TC = cS; \; 0 < S \leq U_1$$

where c is the unit variable cost.

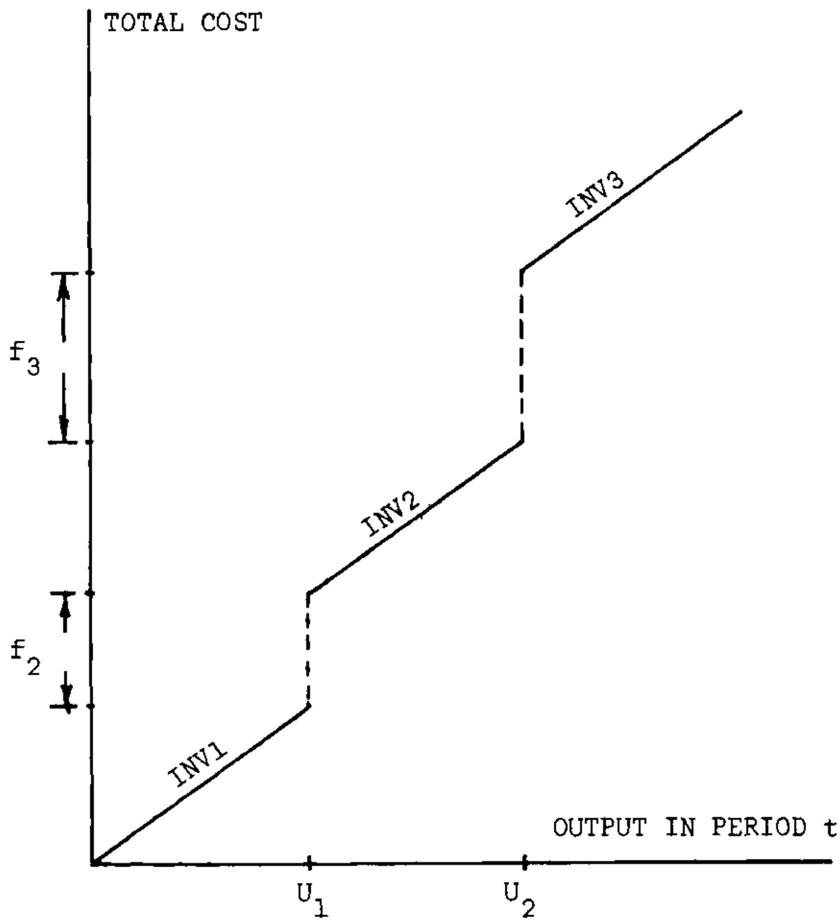

Figure 3.5 Total Cost vs. Output of a Plant p During a Period, When Variable Cost Does Not Change

When an output, S, greater than U_1 but lower than U_2 is required, an additional investment is made with a fixed cost (present value) equal to f_2. The variable cost remains the same as above. Thus, the total cost in this plant for a given period is:

$$TC = f_2 + cS: \quad U_1 < S \leq U_2$$

Similarly for an output greater than U_2 but lower than U_3, the total cost in this period is given by:

$$TC = f_3 + cS: \quad U_2 < S \leq U_3$$

Thus, in order to minimize the cost of a system in this time period we set the following equations:

$$\text{minimize } TC = cS + f_2X_2 + f_3X_3 + 0T$$

subject to capacity constraint of plant p

$$S \leq U_1 + U_2X_2 + U_3X_3$$

or

$$S - U_2X_2 - U_3X_3 \leq U_1$$

capacity constraints for all plants;

$$\phi_i = 0; \; i = k+1,\ldots,I,$$

demand constraints for all products in all markets;

$$\phi'_{mr} = 0; \; m = m+1,\ldots,I, \quad r = 1,\ldots,R$$

integer constraints;

$$X_2, X_3 = 0, 1$$

non-negativity constraints;

$$S \geq 0 \text{ (other variables} \geq 0)$$

Where OT represents other terms in the objective function, such as profit and cost for other plants in the system, ϕ_i are the capacity constraints of plant i and ϕ'_{mr} are the demand constraints per each product m and each market r.

Note that the capacity constraint for this plant and the initial terms of the objective function TC have the same form as the capacity constraints and investments costs, respectively, included in Models A, B, and C.

The assumption that the variable costs are independent of the level of investments or size of the plant is not always true. A new investment may reduce the unit variable cost of operation (economies of scale), or perhaps, it may increase with greater capacity (diseconomies of scale). These two cases will be studied separately below.

3.8.1 Decreasing Variable Costs with New Investments

a) No Discontinuities in the Total Cost Function

The simplest case of decreasing variable costs occurs when the cost function (here assumed piecewise linear) is non-increasing and there are no "jumps." For example, in Figure 3.6 the total cost line for new investments (INV2) intercepts the total cost of the present investments (INV1) at an output less than the upper bound (U_1) of the present capacity of one plant.

Similarly, the interception of line INV2 with INV3 falls in the operating range of the facilities defined by INV2. This situation appears when a substitution of equipment is made or when the capacity of present facilities may be expanded without additional fixed costs.

The total cost function of the system to be minimized is:

$$TC = c_1S_1 + (f_2X_2 + c_2S_2) + (f_3X_3 + c_3S_3) + OT$$

subject to:

$$0 \le S_1 \le U_1'X_1$$

$$0 \le S_2 \le (U_2' - U_1')X_2$$

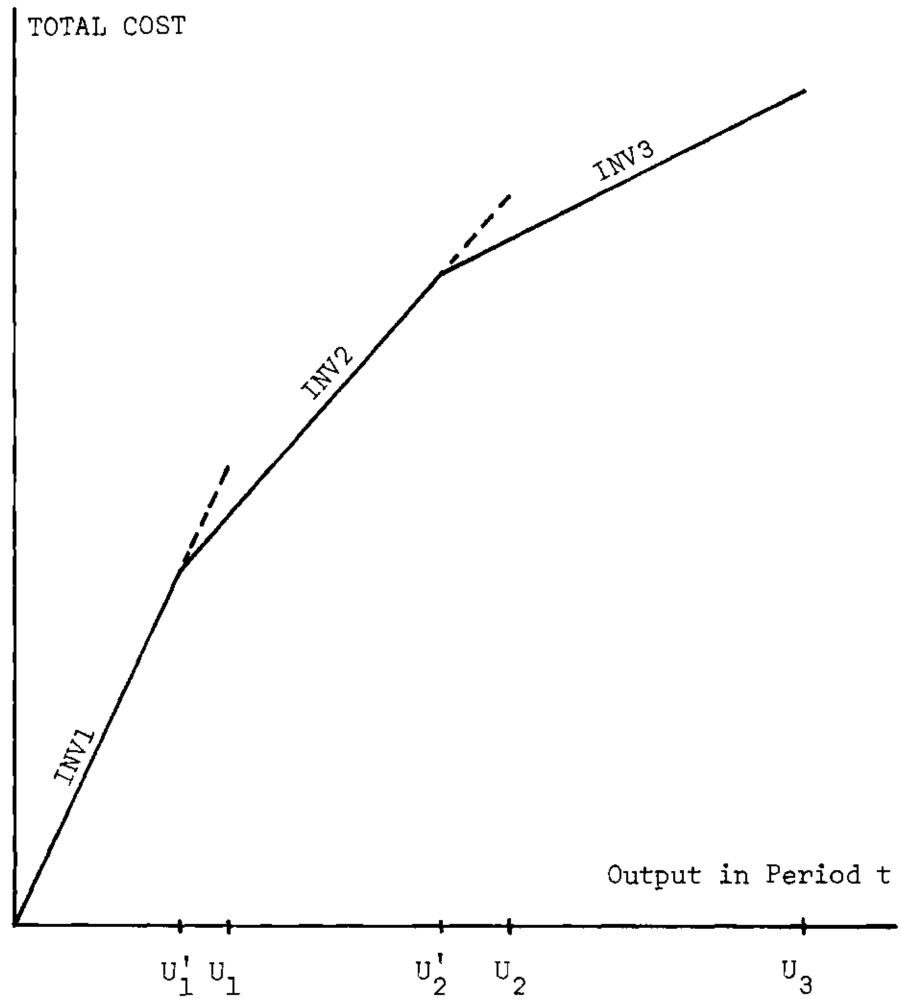

Figure 3.6 Total Cost vs. Output of a Plant in a Period t, Having Decreasing Variable Costs Without Jumps

$$0 \leq S_3 \leq (U_3' - U_2')X_3$$

The following equation guarantees that only one interval will be selected:

$$X_1 + X_2 + X_3 = 1; \; X_1, X_2, X_3 = 0 \text{ or } 1$$

The term OT represents additional income and costs which are explained in the discussion of Model D.

Other inequalities, such as capacity and market constraints for other plants and products, also must be included here. Of course, the non-negativity set of constraints are implicit in the model. One may write the equation $S = S_0 + S_1 + S_2$ to have in one variable S the output per period, regardless of what investments are chosen.

b) Total Cost Function with Discontinuities

The more general situation occurs when there are jumps in the total cost function. In this case, a fixed charge is required in order to increase the present capacity of a plant and reduce its variable cost. Figure 3.7 shows the total cost for a plant in a period. If an output greater than U_1, the upper bound capacity of the present facilities, is required, then an investment with charge f_2 will increase the capacity of the plant up to U_2 and at the same time reduce the unit variable cost from c_1 to c_2.

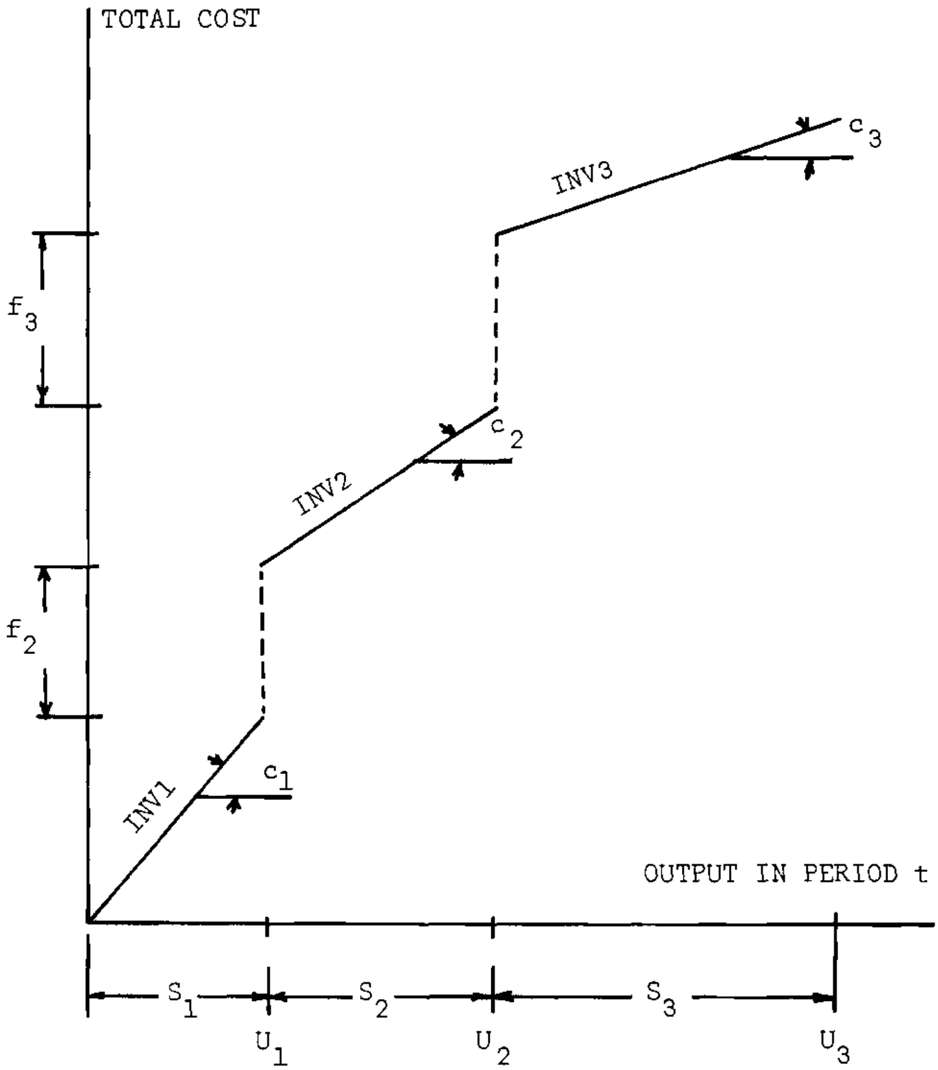

Figure 3.7 Total Cost vs. Output of a Plant in a Period Having Decreasing Variable Cost with Fixed Charges in Each Investment

The total cost TC for an output $0 \le S_1 \le U_1$ is simply $TC = c_1 S_1$. For an output S, $U_1 \le S \le U_2$, the plant total cost function to be minimized is:

$$TC = c_1 S_1 + (f_2 X_2 + c_2 S_2)$$

subject to:

$$0 \le S_1 \le U_1$$

$$S_1 \le U_1 X_2$$

$$0 \le S_2 \le (U_2 - U_1) X_2$$

$$X_2 = 0 \text{ or } 1.$$

If S is the total output of the plant, i.e., $S = S_1 + S_2$, then the total cost may be expressed as $TC = c_1 S + f_1 X_2 + (c_2 - c_1) S_2$.

Similarly, for a system of plants, one of which, say plant p, has the cost pattern shown in Figure 3.7, the model for a single period would be:

$$\text{minimize} \qquad TC = c_1 S + f_2 X_2 + (c_2 - c_1) S_2$$

$$+ f_3 X_3 + (c_3 - c_1) S_3 + OT$$

subject to: $S_1 \leq U_1$

$$S_2 \leq (U_2 - U_1)X_2$$

$$S_3 \leq (U_3 - U_2)X_3$$

$$S = S_1 + S_2 + S_3$$

$$\phi_i = 0; \; i = 1,\ldots,I$$

$$\phi'_{mr} = 0; \; m = 1,\ldots,M$$
$$r = 1,\ldots,R$$

In order to guarantee that $S_2 > 0$ only if $S = U_1$, and that $S_3 > 0$ only if $S = U_2$, the following constraints are added:

$$S_1 \geq U_1X_2$$

$$S_2 \geq (U_2 - U_1)X_3$$

$$X_2, X_3 = 0, 1$$

The term OT stands for the income and cost of other plants, ϕ_1 represents the capacity constraints of the system, and ϕ_{mr} represents the market restrictions for each product in each market.

An example of the application of this model is the case of buying extra facilities which make the present ones more efficient (economies of scale).

3.8.2 Increasing Variable Costs with New Investments

It is possible that an increase in capacity will result in an increase in variable costs (diseconomies of scale). Here, again, two cases will be discussed. In the first, the total cost of a plant is represented by a broken line, without "saltus." The second case describes the situation in which a fixed cost is added to the system whenever an increase in the plant capacity is made, creating jumps in the total cost function.

Increasing variable costs may appear when an addition in plant capacity is obtained by paying premium costs. Some examples are (1) overtime labor cost premiums and (2) equipment rental costs. In such cases, fixed charges may or may not exist.

a) Increasing Costs Without Discontinuities*

The total cost vs. output of a plant having increased costs without jumps is shown in Figure 3.8.

The working region is divided into three sections: Define $0 \le S_1 \le U_1$, $0 \le S_2 \le U_2 - U_1$, and $0 \le S_3 \le U_3 - U_2$, where U_1 is the upper bound of the capacity of the present facilities, U_2 is an upper bound of the capacity when an additional increment of $(U_2 - U_1)$ has been added, etc.

---

*The type of models described here are also used as an approximation in problems of a class called separable convex programming.

The slope of each segment line represents variable costs, and $c_1 < c_2 < c_3$. Because of this property, only one integer 0-1 variable is required in this model. Thus, a system with a plant or process which has a total cost function as in Figure 3.8 can be modelled as follows:

minimize $\qquad TC = c_1S_1 + c_2S_2 + c_3S_3 + OT$

subject to $\qquad 0 \le S_1 \le U_1$

$$0 \le S_2 \le U_2 - U_1$$

and

$$S_1 + S_2 + S_3 \le U_3X; \quad X = 0 \text{ or } 1$$

plus $\qquad$ other constraints concerning other plants in the system and demands for each product in each region.

The term OT represents other terms in the objective function, such as the income, variable cost, and delivery cost, for each product to be manufactured by the system.

If, in the optimal solution, this plant is to be in operation, the integer variable X should take the unit value. The term U_3X defines the upper bound in the plant capacity. Because of increasing variable costs, the method of solution will select that interval in

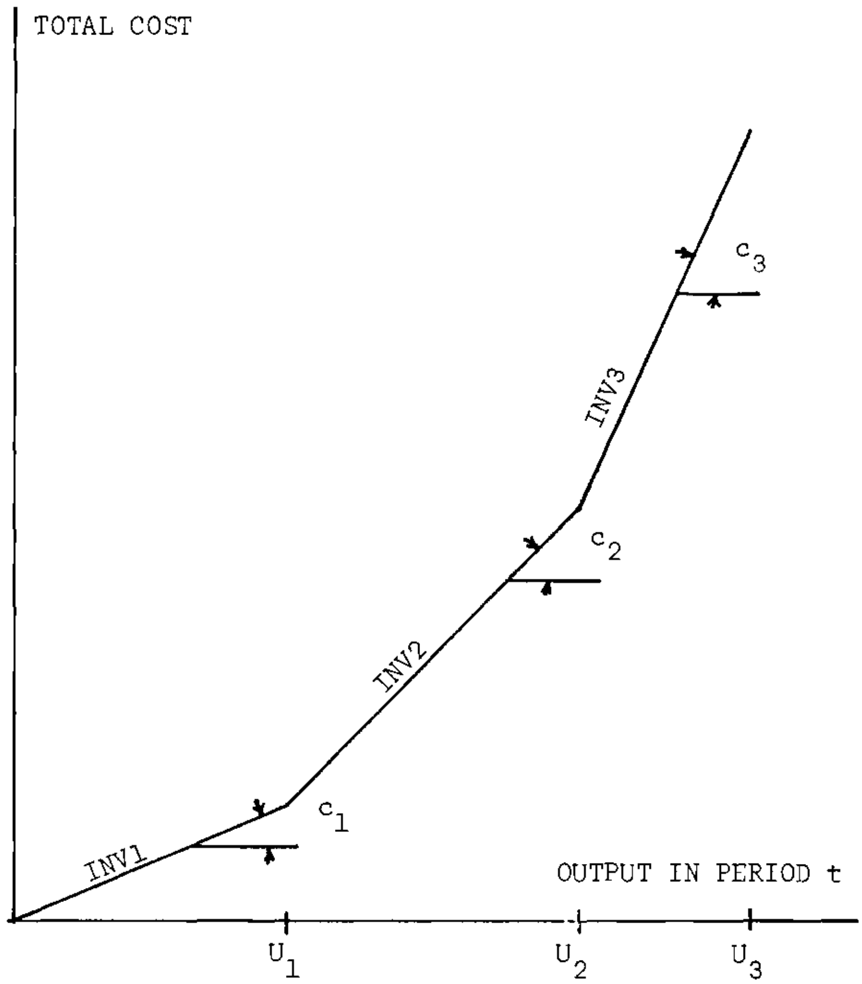

Figure 3.8 Total Cost vs. Output of a Plant Having Increasing Variable Costs Without Discontinuities

which the contribution of the plant to the system is optimal. Thus, only one integer variable is required for each plant having a similar pattern to that shown in Figure 3.8.

b) Total Cost Function with Discontinuities

Increasing the capacity in one plant not only may increase the variable cost, but also may add a fixed cost to the system. Figure 3.9 shows this situation for one plant in the system. In order to have a capacity greater than the present upper bound U_1, a fixed cost f_1 is required. At the same time, an increase in the variable cost (slope) from c_1 to c_2 is observed.

Defining, again, the output of a plant for each interval as $0 \leq S_i \leq U_i - U_{i-1}$, for $i = 1,2,3$ where $U_o = 0$, the model can be written as

minimize $\quad TC = c_1S_1 + (f_2X_2 + c_2S_2) + (f_3X_3 + c_3S_3) + OT$

subject to the total output $\quad S = S_1 + S_2 + S_3 \leq U_1 + (U_2-U_1)X_2 + (U_3-U_2)X_3$

and $\quad X_2, X_3 = 0, 1$

$$0 \leq S_1 \leq U_1$$

$$0 \leq S_2 \leq (U_2-U_1)X_2$$

$$0 \leq S_3 \leq (U_3-U_2)X_3$$

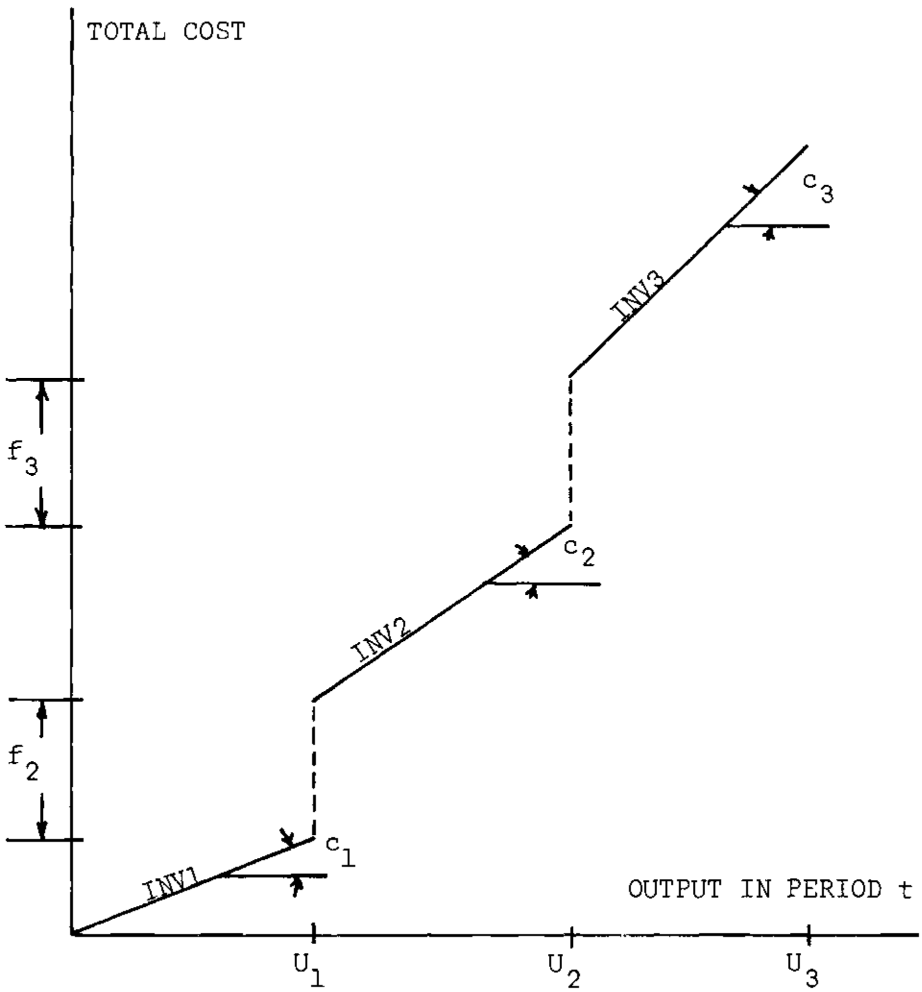

Figure 3.9 Total Cost vs. Output in a Period of a Plant Having Increasing Variable Cost with Charges in Each Investment

plus other constraints of the system.

The term OT represents the income and cost of all activities in the system. Other constraints to be included are the capacity constraints for each plant, the demand for each product in each market, and other pertinent restrictions.

If $U_1 > (U_2 - U_1) > (U_3 - U_2) > \ldots$, then the last constraint in the above model, which defines the upper bound on S_3, is redundant. Note that the set of constraints of the above model does not require that INV3 can be made only if INV2 is selected previously. Suppose, for example, that $(U_3-U_2) > (U_2-U_1)$, that is, the increase in capacity by investment INV3 is greater than the increase corresponding to INV2. In that case, there exists a region where INV1 and INV3 give a lower cost, without requiring the investment INV2. This region $(U_3'-U_2)$ is shown in Figure 3.10.

For example, if the system will need an output $U_2 \lesssim S \leq U_3'$ from this plant, then the total cost of the systems with investments INV3 and INV1 is less than the cost of INV1 + INV2 + INV3. But if the plant output is greater than U_3', then a sequence of investments INV1, INV2, and INV3 will be the best choice.

If, for technological reasons, the investment INV3 requires investment INV2, the additional constraint $X_3 \leq X_2$ should be included in the model. (If $X_2 = 0$, then $X_3 = 0$; but if $X_2 = 1$, then $X_2 = 0$ or 1).

The situation in which there are some plants in the system with decreasing variable costs and saltus in the total cost function (see Section 3.6.1-a) is presented in the following section. Other cases may be adapted from the discussion just presented.

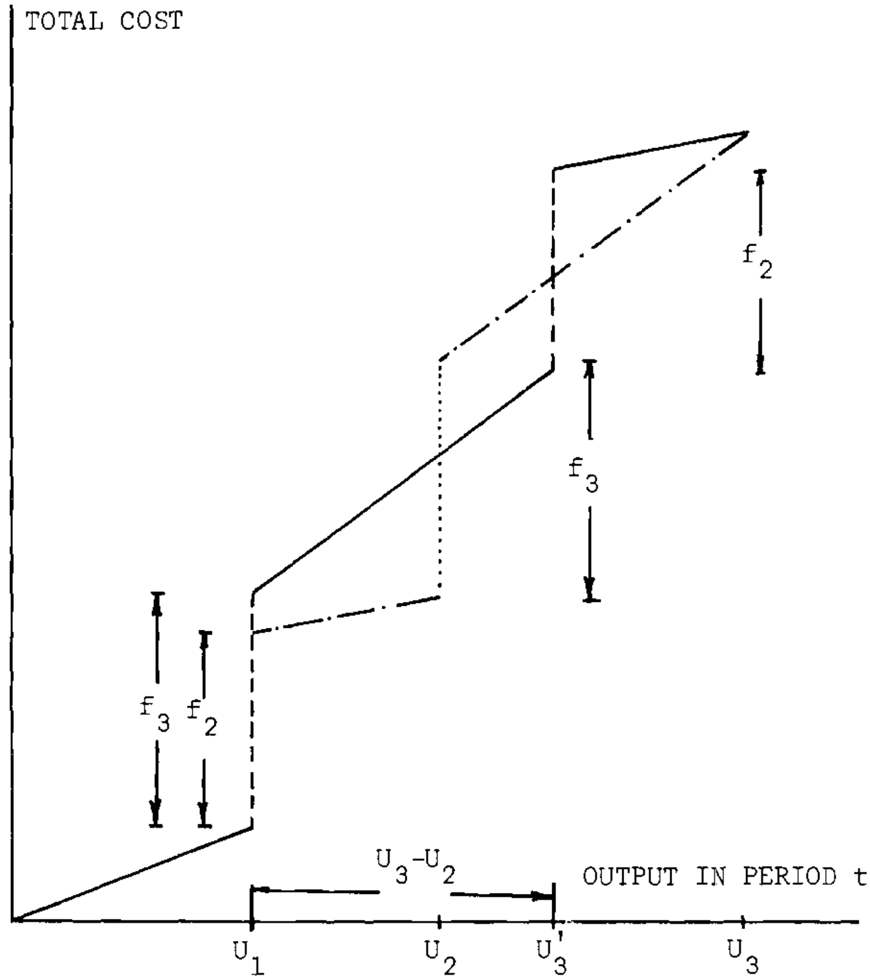

Figure 3.10 Total Cost vs. Output of a Plant. Sequence of Investments

a) INV1, INV2, and INV3
b) INV1 and INV3

3.8 Model D: Multi-Period with Decreasing Variable Costs and Saltus in the Total Cost Function

Assumptions

1-2. Assumptions 1 and 2 from Model B, Section 3.5, hold.

3. The variable cost in some plants ($p = k+1,\ldots,I'$; where $I' \le I$) will decrease at the time when new investments are made. Also, each increase in capacity incurs a fixed cost.

Parameters

Same as Model B plus

$UB(S)$ = upper bound on the variable S. The variable S represents output of a plant or a process under given conditions

$c^t_{mp(j)v_p}$ = new variable cost of producing one unit of product in plant p, process j, during period t, if the project v_p is chosen

$\beta^t_{p(j)v_p}$ = additional capacity added to plant p, process j, at the beginning of period t, if project v_p is selected

$\Delta c^t_{mp(j)v_p} = c^t_{mp(j)v_p} - c^t_{mp(j),v_p-1}$

= difference in variable cost by changing the investment in plant p from project $v_p - 1$ to project v_p. Note that this quantity is non-positive. The present facilities at time $t = 0$ are considered as project $v_p = 1$.

Variables

Same as Model B plus

$S^t_{mp(j)v_p}$ = additional units of production produced at plant p, process j, during time period t, if project v_p is chosen. It is evident that

$$a^t_{mp(j)} S^t_{mp(j)v_p} \le \beta^t_{p(j)v_p}$$

Also,

$$S^t_{mpr} \le \sum_{j=1}^{J_p} \sum_{v_p=1}^{V_p} S_{mp(j)v_p}$$

Objective Function

Assume that a plant p has V_p projects to be undertaken at the beginning of period of time t. When a new addition to capacity (i.e., a new project) is chosen, a fixed cost ($g^t_{v_p}$) is charged, but at the same time the variable cost decreases. The total cost function is similar to that shown in Figure 3.11 for $V_p = 3$. All subscripts and superscripts of the variables in the figure have been eliminated, except for v_p. When the output of this plant is constrained to be less than or equal to UB(S), no change is required in the objective function of Model B because it already contains the cost $c_{mpr}S_{mpr}$.

If additions in capacity are permitted in plant p, the cost of its V_p projects during period t is:

$$\sum_{v_p=1}^{V_p} \eta_{v_p} X^t_{v_p} + c^t_{mp(j)1} S^t_{mpr}$$

$$+ \sum_{v_p=2}^{V_p} D^t c^t_{mp(j)v_p} S^t_{mp(j)v_p}$$

However, since the first two terms are already included in Z_B, the only term to be added is the third term, with summations over plants and over periods.

Then, the objective function of Model D becomes:

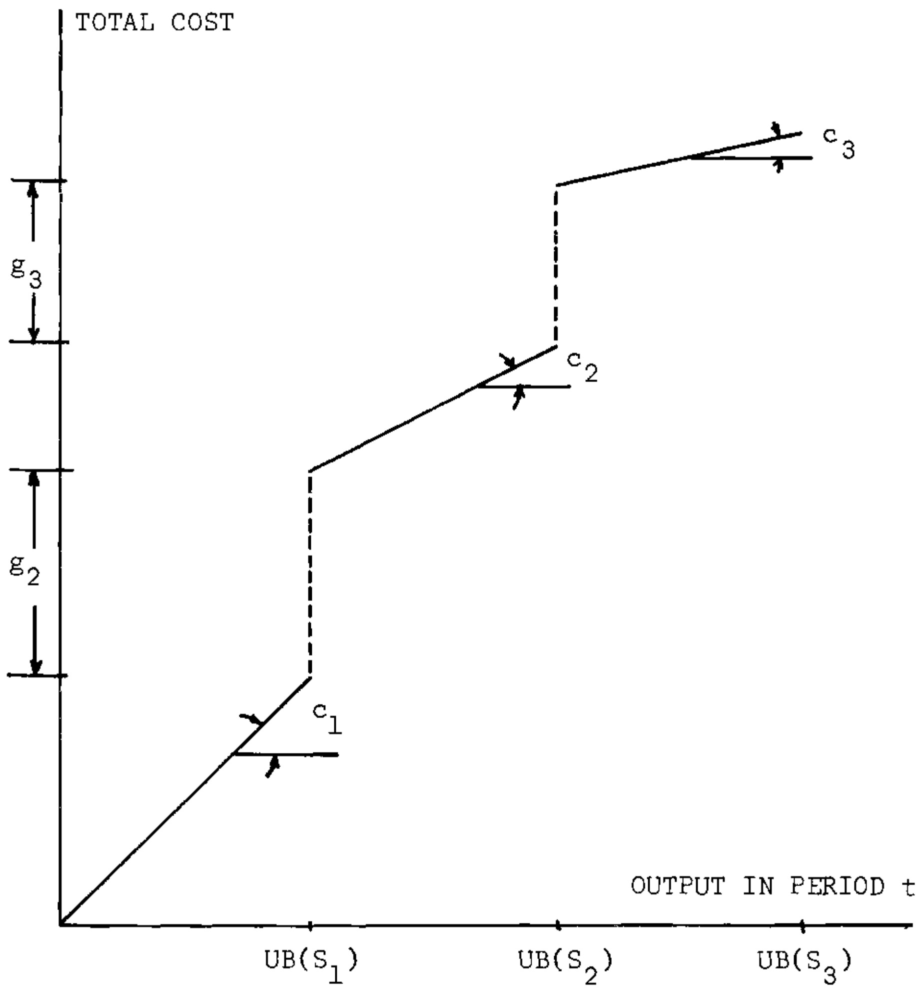

Figure 3.11 Total Cost vs. Output of Plant p, During Period t

$$Z_D = Z_B - \sum_{t=1}^{T} \sum_{m=1}^{M} \sum_{p=k+1}^{I'} \sum_{j=1}^{J_p} \sum_{v_p=1}^{V_p} \Delta c^t_{mp(j)v_p} S^t_{mp(j)v_p}$$

<u>Constraints</u>

Same as Model B plus

$$\sum_{r=1}^{R} S^t_{mpr} \leq \sum_{j=1}^{J_p} \sum_{v_p=1}^{V_p} S^t_{mp(j)v_p}$$

Units of product m sent from plant p to all markets during time t	Total output of product m, manufactured at plant p during time t

for $m = 1,\ldots,M$; $p = k+1,\ldots,I''$; $t = 1,\ldots,T$.

$$\sum_{m=1}^{M} S^t_{mp(j)v_p} \leq \beta^t_{p(j)v_p} X_{v_p}$$

for $v_p = 1,\ldots,V_p$; $p = k+1,\ldots,I$; $t = 1,\ldots,T$.

$$X_{v_p} \leq X_{v_{p-1}};$$

for $v_p = 2,\ldots,v_p$.

$$S^t_{mp(j)v_p-1} \leq UB(S^t_{mp(j)v_p})X_{v_p}$$

for $v_p = 2,\ldots,V_p$; $p = k+1,\ldots,I$.

The last two sets of constraints guarantee that a new project is selected when the required capacity of the plant exceeds the present upper bound.

Other models (including inventory, for example) may be written as a combination of the models presented here.

CHAPTER IV

METHOD OF SOLUTION

4.1 Introduction

In Chapter III, different models describing several situations in expansion planning were presented. A general characteristic belonging to all of them is that some variables may be continuous and others can take only the values zero or one. These models fit in the framework of mixed-integer programming.

If I is a vector of integer variables and X is a vector of continuous variables, the mixed-integer problem can be written as follows:

(4.1) Minimize $z_1 = c_1 I + c_2 X$

Subject to $A_1 I + A_2 X \geq b$

$I, X \geq 0$; I integer

The difficulty of solving problems when some or all of the variables must be integers has been recognized since the beginning of mathematical programming theory [18]. Much research is still under way, and only in recent years have methods been available for the solution of such problems.

There are presently three different approaches to solving mixed-integer problems. They are: a) cutting planes, b) partitioning and, c) branch and bound. None of them is suitable for all mixed-integer programming problems. The first two will be described briefly, while the branch and bound approach will be described in some detail.

The fact that the number of continuous variables in all models is much greater than the number of discrete variables, and that the latter variables are low-leveled (i.e., only two values are allowed, 0 or 1), suggests the use of a branch and bound algorithm to solve the expansion planning problem.

Since pure-integer programming seems to be easier than mixed-integer programming, methods of changing a mixed problem to a pure-integer problem are important. In particular, if the number of integer variables in the problem is moderately large while the number of continuous variables is small, the partitioning approach may be a suitable method to solve this type of problem.

4.2 Cutting Planes

The idea of cutting planes was used for the first time in 1954 by Dantzig, et al. [18], and later by Markowitz and Manne [19]. However, their application of cutting planes was unsystematic and, more important, convergence was not proved. The first finite general methods to solve all-integer and mixed-integer problem were advanced by Gomory [20].

The cutting plane method starts with the optimal solution of problem (4.1) without considering the integer constraints. The convex hull defined by the constraints H, $A_1I + A_2X \le b$ (I, X $\ge$ 0; I not restricted to be integer) contains all points of the feasible set. If one finds an optimal solution which is not feasible, then an additional linear inequality or cutting plane is generated and added to H. This new constraint is not satisfied at the current non-integer solution, but must be satisfied by all feasible integer points. Thus, the new cutting plane and the original constraints define a new convex hull which also contains all feasible integer points. The resulting linear programming problem is solved, and if integer requirements are not met, an additional constraint is added. This procedure is continued until the integer variables are forced to take integer values. The problem is to generate the new constraints in such a way that the mixed-integer solution is obtained without excessive computational time.

One advantage of the method is that it uses the pivoting rules of the simplex method to move from one extreme point to another on the convex hull. Gomory proved that if a mixed-integer solution exists, it can be obtained in a finite number of iterations. Reports of applications, however, indicate that convergence can be extremely slow. Beale reported that it has been shown that unless the objective function itself is integral, the method may not converge [21].

4.3 Partitioning

Another approach for solving mixed-integer problems was suggested by Benders [22]. The rationale behind the method is to convert the original mixed-integer problem into an all-integer problem, and then to solve it by using an all-integer algorithm. This method is suitable when the number of integer variables is larger than the number of continuous variables, as will be seen below.

A very inefficient method of solving problem (4.1) is to list all possible vectors I; then, using each of the possible values for I, solve the continuous part as a regular linear programming problem. Thus, problem (4.1) becomes:

(4.1a) Minimize $z_1 = c_1 I + c_2 X$

Subject to $A_2 X \geq b - A_1 I$

$X \geq 0$

Note that once vector I is defined, the term $c_1 I$ is constant. For a given vector I, say I_j, the remaining continuous part of problem (4.1) may be expressed as

(4.3) Minimize $z_2 = c_2 X$

Subject to $A_2 X \geq b - A_1 I_j$

$X \geq 0$

Let X_j^* be the optimal solution to the above problem corresponding to the I_j vector. If problem (4.1) has a feasible solution, its optimal solution is that in which $c_1 I_j + c_2 X_j^*$ is minimum. That is, $z_1^* = \min_j \{c_1 I_j + c_2 X_j^*\}$.

The method is inefficient because we need to enumerate all possible integer vectors I_j before knowing if we have the optimal solutions.

The dual of problem (4.1a) is stated as follows:

(4.3) Maximize
$$z_2 = W(b - A_1 I)$$
$$WA_2 \le c_2^T$$
$$W \ge 0$$

where W is the row vector corresponding to the dual variables, I is a given integer vector, and superscript T means transpose.

The original problem (4.1) can now be written:

(4.4) Minimize
$$z_1 = c_1 I + \max_W \{W(b-A_1 I)\}$$
Subject to
$$WA_2 \le c_2^T$$
$$W \ge 0$$

With this new reformulation, we now have a convex set, ω, $\omega = \{W | WA_2 \le c_2^T;\ W \ge 0\}$ independent of the integer vector I. If ω

is empty, then there is no solution to the original problem. Otherwise, the $\max_{W}\{W(b-A_1)\}$ occurs at an extreme point of ω or grows unbounded along an extreme ray of ω. The convex set ω contains a finite number of extreme points and extreme rays, which can be enumerated. If the set ω does not contain any extreme rays, that is, ω is strictly bounded, one can enumerate all extreme points, W^k in ω, and solve a new integer problem,

(4.5) Minimize $\quad z$

$$\text{Subject to} \qquad z \geq c_1 I + \max\{W^k(b-A_1 I)\}, \text{ for all } W^k \varepsilon \omega$$

$$W^k \varepsilon = \omega$$

$$I \geq 0 \text{ and integer}$$

Recall that W^k is an extreme point of the set of constraints (4.4) and has been previously specified.

Note, also, that each extreme point W^k generates one constraint in (4.5). To find the optimal solution of problem (4.1), or equivalently to (4.5), we have to include all extreme points $W^k \varepsilon \omega$, which could represent a formidale amount of work. Bender's algorithm uses only a subset of the extreme points.

Before presenting the algorithm, we write the general case, i.e., when the set ω contains both extreme points W^k and extreme rays W^r. Let R be the set of extreme rays, W^r, $R \subset \omega$.

If, for some I, there exists a W^r, $r \in R$ such that $W^r(b-A_1I) > 0$, then the maximum part of (4.4) becomes $+\infty$, and the minimization problem is not feasible. The existence of this unbounded solution to problem (4.3) implies that for this given integer vector I, problem (4.2) is infeasible. Therefore, we need to add the following restriction to the integer vector I,

$$W^r(b-A_1I) \leq 0, \quad \text{for all } W^r \in R$$

Problem (4.5) may be rewritten as:

(4.6) Minimize $\quad z$

Subject to $\quad z \geq c_1I + \max_{W^k \in \omega}\{W^k(b-A_1I)\}$

and $\quad W^r(b-A_1I) \leq 0, \text{ for all } W^r \in R$

$$I \geq 0 \text{ and integer}$$

Again, in order to find the optimal solution (4.1), it is necessary to enumerate all extreme points W^k and extreme rays W^r continued in ω, include them in (4.6), and then solve the resulting integer program.

However, this is not necessary in Bender's algorithm, because it uses only a subset of extreme points and extreme rays in order to decrease the number of constraints to be included in (4.6).

The algorithm consists of the following steps:

a) Select a subset of extreme points and extreme rays obtained in (4.4) and put them in (4.6). Call this subset Q.

b) Solve the resulting integer programming problem (4.6).

c) If no feasible solution exists to the partial problem, then no feasible solution exists to the original problem. Terminate.

Let z^*, I^* be the optimal feasible solution obtained in step b). If the problem is unbounded below, then z^* takes a small value.

d) Solve the dual problem (4.3), using $I = I^*$.

If $z_2^* - \infty$ (unbounded solution), then the actual integer solution I^* does not permit a feasible X. Then a new extreme ray W^r, $r \in R$ has been found, namely W^*, and it is added to the current subset of extreme points and extreme rays Q; go to step b).

If $z_2^* = -\infty$, then no feasible solution exists. Terminate.

If z_2^* is finite, call the solution vector W^* and the corresponding primal vector in (4.2) X^*, both with an objective function equal to z_2^*. Then a feasible solution (I^*, X^*) in terms of the original variables has been obtained and its value is z_2^*.

e) In order to test optimality, make the comparisons: If $z^* \geq c_1 I^* + z_2^* = c_1 I^* + c_2 X^*$, we have an optimal solution (I^*, X^*) of the original problem. Terminate.

But if $z^* < c_1 I + z_2^*$, then the extreme point W^* does not satisfy the set of constraints of (4.6). In particular, the constraint $z \geq c_1 I + W^*(b - A_1 I)$ of (4.6) is violated. Therefore, this new constraint is added to the subset Q; go to step (b).

Obviously, the method converges, because at most we have to enumerate and work with a finite set of extreme points and extreme rays. Some advantages of this method are as follows:

a) In each step we may have a feasible, not necessarily optimal, solution. This is a characteristic found in primal methods.

b) In each step we have a lower bound and an upper bound of the objective function z_1. A lower bound is given by z^* since problem (4.5) in step b) is solved with a subset of constraints. An upper bound is found in step d), and is z_2^* (which may be infinite).

d) The above advantages permit stopping the algorithm at any time after having a feasible solution. The bounds give us the range in which the optimum lies.

Although Gomory's and Bender's algorithms may be used to solve some classes of mixed-integer problems, other approaches exist which may be more efficient in solving the 0-1 mixed problem with a few integer variables. This approach is called branch and bound.

4.4 Branch and Bound. The Land and Doig Algorithm

The first algorithm which presented the branch and bound idea appeared in 1960; it was developed to solve mixed-integer problems [23]. Although the authors, Land and Doig, did not call it such, this paper was a pioneer effort in a new approach to solve a diversity of problems. This type of implicit enumeration was called branch and bound for the first time by Little, et al. [24] in their algorithm to solve a traveling salesman problem.

Branch and Bound is an intelligent method of searching the space of feasible solutions to a given problem. The branching step partitions the space of feasible solutions into mutually exclusive subsets (branches). Associated with each branch is a value of the objective function which represents a lower bound (in a minimization problem) on all problems which originated from that branch. The branch with lowest cost is selected for further partitioning. Other branches with greater cost are not considered for branching at this stage, but may be reconsidered at a later stage. The branching procedure continues until a feasible solution is obtained (assuming it exists) in one branch, while the other branches have a greater lower bound, whether their solution is feasible or not.

The Land and Doig Algorithm

In a few words, the Land and Doig method involves the construction of a tree of linear programming problems, the root of the tree being the original problem (4.1) without integer constraints.

The modified problem is:

(4.7) Minimize $Z = c_1 I + c_2 X$

Subject to $A_1 I + A_2 X \leq b$

$I, X \geq 0$

If the optimal solution to the modified problem given integer values to those variables which are restricted to be integers, then this is also the optimal solution to the original problem (4.1). Otherwise, the optimal value of the objective function of (4.7) is a lower bound to the mixed-integer problem.

If the integer variable x_k appears with a non-integer value in the optimal solution of the modified problem, then two linear programming problems are constructed by setting x_k equal to the next lower integer value and to the next higher integer value, respectively.

Both problems are solved, and the branching is repeated again in that problem with lower optimal objective function (in a minimization problem). The process continues until the optimal solution of a problem gives integer values to all variables restricted to be integers, and its objective function is a lower bound on all candidate branches. With this method, possible solutions are enumerated in such a way that only those which could have the optimal solution are considered. This is an implicit enumeration, rather than a complete enumeration.

The tree is made up of branches and nodes. Each branch is associated with an integer constraint of the type: x_k = a constant. Each node identifies the number of the linear programming problem constructed with the original constraints in (4.7) plus all integer constraints represented by the branches connecting this node with the node 0. Problem (4.7) corresponds to node 0. The optimal objective function of a problem is associated with its node. Thus, if "n" is a node, then z_n^* is the optimal value of functional of the nth problem.

Figure 4.1 shows a tree diagram for a mixed-integer problem solved by the Land and Doig algorithm.

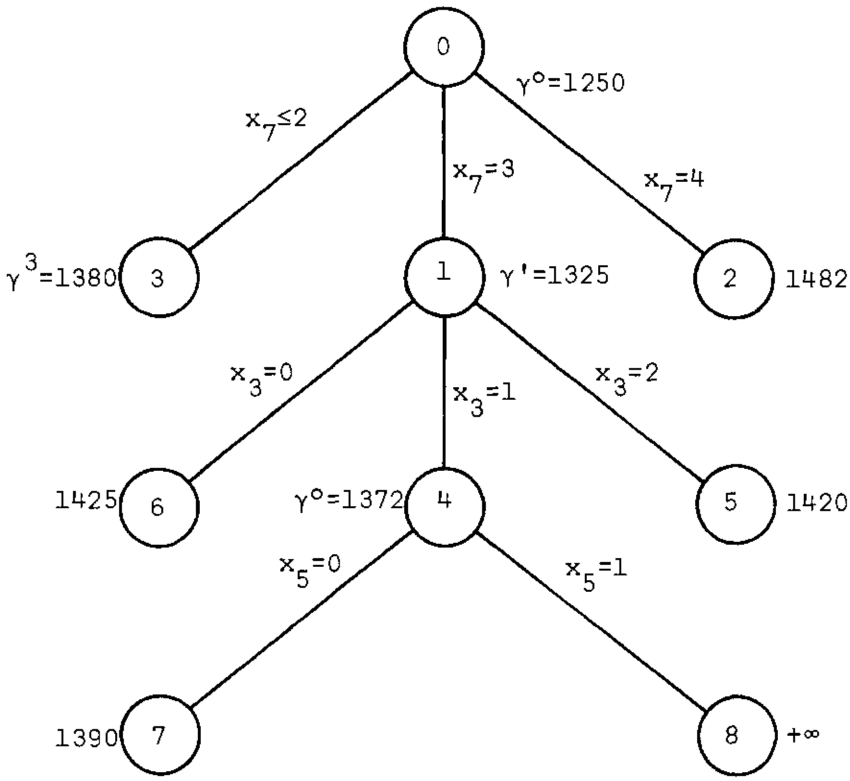

Figure 4.1 A Tree Diagram for the Solution of a Mixed-Integer Problem by Branch and Bound

Thus, the node (5) means that the optimal objective function of the original problem plus two constraints, $x_7 = 3$ and $x_3 = 2$, is equal to 1420. The node (8) has a functional equal to $+\infty$, which indicates that this solution is not feasible.

The algorithm is described in more detail below, but we first must define J as the set of indexes assigned to those variables restricted to be integers, i.e., $J = \{j | x_j \text{ is integer}\}$, and γ^k as the least lower bound on the functional of problem (4.7), at the kth stage. Now the steps are as follows:

a) Solve the original problem without integer constraints (4.7). If the solution gives x_k = integer, for all $k \in J$, then this is the optimal solution to (4.1). If not, assign the functional to the node (0), with value γ^o. (In the Figure 3.1, $\gamma^o = z_o^* = 1250$.)

b) Select a basic variable x_k, $k \in J$, which has a value furthest from an integer. Let this variable be x_r, $r \in J$. Then construct two branches, one with $x_r = v = [x_r]$ and the other with $x_r = v' = v + 1$. The two additional linear programming problems are solved, and the respective optimal functionals assigned to nodes (1) and (2). ($z_1^* = 1325$, $z_2^* = 1482$).

c) If lower bounds on the objective function $0,1,\ldots,k-1$ have already been assigned, then choose the node with the lowest functional, and assign this value to γ^K. (In the example, $\gamma^1 = \min(z_1^*, z_2^*) = 1325$.)

d) If the node with least lower bound γ^K has all x_k, $k \in J$, with non-negative integer values, then this is an optimal solution to (4.1). Terminate. Otherwise, find the node q just above the node with least lower bound. Let $x_s = w$ be the branch joining these two nodes ($\gamma^q = \gamma^o$, $x_s = x_7$, $w = 3$).

e) From the node q, draw two branches with associated constraints $x_s \leq w - 1$ and $x_s = w + 1$. (Note that one of these branches has already been in the tree.) Solve the problems and select that node with minimum functional as a second choice for a branch from node q. (The branches are $x_7 \leq 2$ and $x_7 = 4$; the latter being already in the tree.)

f) Two new branches are drawn from the node with the least lower bound γ^K, by setting the x_k, $k \in J$, with the value furthest from an integer to its neighbor integer values, and the respective linear problems are solved. The new values of the objective function for this problem are greater than or equal to γ^K. If one of these problems is not feasible, this branch is closed and investigated no further. If neither of these branches has a feasible solution, then the original problem (4.1) has no feasible solution. (The branches are $x_3 = 1$ and $x_3 = 2$, with $z_4^* = 1375$ and $z_5^* = 1420$.)

g) At this point three new branches have been generated. Go to step c). (The three branches are, $x_7 \leq 2$, $x_3 = 1$, $x_3 = 2$.)

Since the sequence of $\gamma^o, \gamma^1, \ldots, \gamma^K, \ldots$, gives nondecreasing lower bounds on the value of the objective function of the original problem (4.1) and the method used to add nodes and arcs covers all possibilities, then the algorithm will find the optimal solution, if it exists, or will indicate an infeasible solution.

The Land and Doig algorithm has led to new algorithms for the solution of mixed-integer problems, nonlinear programming problems, plant location problems, etc. In particular, the mixed-integer algorithms by Driebeek and Rebelin, described below, are based on the ideas advanced by Land and Doig.

4.5 Driebeek's Algorithm [25]

Driebeek's algorithm is designed for the solution of large mixed-integer problems which have a few low-level integer variables.

The algorithm begins with the optimal solution of the modified problem (4.7) where all integer variables have been transformed into a sum of 0-1 variables. A sensitivity analysis is made in the optimal simplex tableau. The results of this analysis are summarized in a penalty table, in which each cell indicates the increase (decrease) of the objective function for a minimization (maximization) problem, when an integer variable is set to a specified integer level. The next step consists of selecting some trial mixed-integer solutions.

In each trial an integer level is specified for *each* integer variable. The penalty table is used to list the trials to be computed, on the basis of being the most promising integer solutions. In order to know if a given trial will have a feasible solution, the right-hand side of the optimal continuous solution is modified. If it is necessary, the dual-simplex algorithm is applied to the new tableaus. Since only the right-hand side is changed, the tableau in any step is always optimal, but may be infeasible.

Each trial begins from the optimal continuous solution of problem (4.7). The functional or objective function of the continuous solution is the lower bound on the mixed-integer solution in a minimization (maximization) problem. Once a feasible mixed-integer solution is found, an upper (lower) bound on the objective function of the original problem is obtained. In such cases, the penalty table is used again to search only those trials whose penalties are less than the difference between the lower and upper bounds. In this way many trials will be discarded from the search.

The author points out three features of the algorithm:

1. The optimal mixed-integer solution is found (convergence).
2. The convergence is quite rapid.
3. Since each trial starts from a common solution, the method is insensible to round-off errors and to degeneracy.

As was stated above, all integer variables of (4.7) are expressed as a sum of 0-1 integer variables. Thus, if $x_k \leq n$, then $x_k = \sum_{j=1}^{n} t_{kj}$ and the restriction becomes $\sum_{j=1}^{n} t_{kj} \leq n$ where $t_{kj} = 0,1$. When x_k takes a value less than the upper bound, say $x_k = j$, $j < n$, it is convenient to have each of the first j terms equal to 1, and the rest (n-j) terms equal to 0. This can easily be done by writing the following conditions, instead of $\sum_j t_{kj} \leq n$

$$1 \geq t_{k1} \geq t_{k2} \dots \geq t_{kn} \geq 0, \tag{4.8}$$

which are equivalent to (n+1) additional constraints and (2n+1) variables (including slack variables). A problem with m integer variables, each having m_i, $(i=1,\ldots,m)$, integer values generates a total of $M = \sum_{i=1}^{m} (m_i+1)$ additional constraints to the modified problem (4.7).

Note that the equation $\sum t_{kj} \le n$ is not written because it is implicitly included in (4.8). For example, if $x_k \le 3$, the following constraints are added to the initial problem.

Constraint	t_{k1}	t_{k2}	t_{k3}	s_{k0}	s_{k1}	s_{k2}	s_{k3}	b
s_{k0}:	1			1				1
s_{k1}:	-1	1			1			0
s_{k2}:		-1	1			1		0
s_{k3}:			-1				1	0

Where s_{kj}, $j=0,\ldots,3$, are the integer slack variables.

During the optimization of problem (4.7) plus the set of constraints defined in (4.8), the integer constraints for variables t_{kj} do not hold. The only requirement is that $0 \le t_{k,j} \le t_{k,j+1} \le 1$ for $j=0,1,\ldots,n-1$, where k denotes that the t variables are part of the integer variable x_k and j denotes the possible integer level which the variable can take.

Once the optimal continuous solution is obtained, the constraints (4.8) are activated by subtracting a unit delta right-hand side vector, Δb, to the right-hand side vector b. For example, if $x_k = 2$, then the Δb vector will have a unit value in the s_{k2} row (third row) and zero elsewhere, $\Delta b = (0,0,1,0)^T$. When Δb is subtracted from b, the resulting vector becomes $b - \Delta b = (1,0,-1,0)^T$, which implies that $t_{k1} = t_{k2} = 1$ and $t_{k3} = s_{k1} = \ldots = s_{k3} = 0$. Thus $x_k = \sum t_{kj} = 1 + 1 + 0 = 2$.

In an article by Zionts [26], a more compact representation is suggested. He shows that the set of (n+1) constraints and (2n+1) variables required for each integer variable x_k, $x_k \leq n$ in Driebeek's algorithm can be reduced to a single constraint and (n+1) variables. If the (n+1) equations which were set up to satisfy the requirements of $0 \leq t_{k,j} \leq t_{k,j+1} \leq 1$ $(j=1,\ldots,n-1)$ are added together, the t variables are eliminated and the resulting equation is

$$(4.9) \qquad s_{k0}\colon \quad s_{i0} + s_{i1} + \ldots + s_{in} = \sum_{j=1}^{n} s_{ij} = 1$$

Since the t's are not present any more, the problem is to get x_k in a function of the slack variables s_{kj}.

If the last (n-j+1) equations from (4.8) are added, the resulting equation becomes:

$$t_{kj} = \sum_{h=j}^{m} s_{kn}$$

and

$$x_k = \sum_{j=1}^{n} t_{kj} = \sum_{j=1}^{m} \left(\sum_{h=j}^{m} s_{kn} \right) = \sum_{j=1}^{m} j s_{kj}$$

or

$$x_k = s_{k1} + 2s_{k2} + \ldots + ms_{kn}$$

For 0-1 variables $x_k = s_{k1}$. However, note that the slack variable s_{k0} is still in the tableau by Equation (4.9) and will be used to calculate penalties.

To set an integer variable into a given level $x_k = j$, it is sufficient to bring s_{kj} into the basis, ignoring as candidates for entry into the base all other slack variables s_{kh}, $h \neq j$. For example, if $x_k = 2$, then s_{k2} is forced to be in the basis (at level one) and s_{k0}, s_{k1} and s_{k3} have to remain as now basic variables. Note that Zionts' procedure does not require the Δb vector, to force the x_k variable to be at a given level.

Sensitivity Analysis and Penalty Calculations

The penalty calculation will be described according to the original paper by Driebeek, and it is assumed that the objective function is to be minimized.

The right-hand side delta vector, Δb, to be introduced at the first tableau, also can be introduced at the final tableau if it is "up-to-date." Thus, $\Delta b^* = B^{-1} \Delta b$ where:

Δb^* is the delta unit vector to be subtracted from b^* at the optimal tableau,

B^{-1} is the inverse matrix corresponding to the vectors on the optimal basis, and

Δb is the original delta unit vector to force a solution to a single lattice point.

There are two cases for calculating the penalties associated with each level of the integer variables. The cases depend upon the status of the slack integer variables, s_{kj}, in the optimal continuous solution of problem (4.7) where constraints (4.8) have been added. Case 1 is concerned with the penalty calculation when the integer slack variable, s_{kj}, is non-basic; Case 2, when the integer slack variable is basic.

<u>Case 1</u>

No basic slack variables. True penalties.

First suppose that not all integer slack variables appear in the optimal continuous solution. The "opportunity costs" for these variables are non-negative in the final tableau. When Δb is subtracted, the optimal objective function z^* is reduced by $c_B \Delta b^*$, where c_B is the price vector associated with the optimal basic variables. Then,

$$\Delta z = c_B \Delta b^* = c_B B^{-1} \Delta b$$

but since the dual vector associated with the inverse matrix is $\pi = c_B B^{-1}$, then $\Delta z = \pi \Delta b$.

This Δz is the minimum decrease in the objective function which will be obtained when a given integer point defined by Δb is forced into the solution. An extra reduction may appear because of infeasibilities created by the subtraction of Δb. Elimination of these infeasibilities by the dual simplex method will further reduce the optimal objective function.

In other words, if the integer nonbasic variables are forced to be in the solution, a decrease in z^* of at least $\pi\Delta b$ will be obtained; if infeasibilities are created, then an additional reduction will be obtained.

The value Δz is then the least decrease in z^* when a specified integer solution is forced into the solution. This is called the "minimum penalty" for this given integer solution.

Since Δb contains only zeroes and ones, the Δz is easily obtained by summing those reduced costs corresponding to the slack variable whose constraint is being activated.

Thus, for those nonbasic slack variables in the optimal table, the true penalties are simply the reduced costs.

Case 2

Basic slack variable. Pseudo penalties.

If one constraint is loosely satisfied, its slack variable will appear at the optimal solution, with a value $0 \le s^*_{kj} \le 1$. If a Δb vector, with 1 at the s_{kj} row, 0's elsewhere, is subtracted in the optimal table, the updated Δb^* will be equal to Δb, since Δb vector is a unit vector. Therefore, the modified s_{kj} in the optimal tableau becomes $s'_{kj} = s^*_{kj} - 1$ or $-1 \le s'_{kj} \le 0$.

Except in the case that $s'_{kj} = 0$, an infeasibility is introduced. In order to break it, the dual simplex method is applied. That is, if the infeasibility occurs at row R, then column K is chosen, such that $d_k/a^*_{RK} = \min(dn/a^*_{Rh}$, for all h with $a^*_{Rh} < 0\}$ where a^*_{Rh} is the value of the a_{Rh} element in the optimal table and d_h is the reduced cost of the hth column, and h is a subscript referring to a column. If no K column can be found, then the problem has no solution when $x_k = j$. The penalty is set to $+\infty$. Otherwise, the penalty, i.e., the reduction on the objective function, will be $s'_{kj}(d_K/a_{RK}) \geq 0$.

The pseudo penalty calculation can be summarized as follows: Find a row where a basic slack variable appears; say s^*_{kj} appears in row R. For this row, select the column with $\min_h\{d_h/a_{Rh};\ a_{Rh} < 0\}$. If no column has $a_{Rh} < 0$, then the pseudo penalty associated with s_{kj} is $+\infty$. If a column is found, say K, then the pseudo penalty is equal to $(s^*_{kj} - 1)(d_K/a_{RK})$. At this point, each basic (nonbasic) variable has associated with it a pseudo (true) penalty.

An integer solution is selected by specifying a level for each integer variable by means of the Δb vector. At the time of obtaining an integer solution, a decrease in the objective function will be incurred in an amount of at least:

1) the sum of the true penalties associated with the nonbasic slack variables or

2) the maximum of the pseudo penalties associated with the basic variables.

With these facts, a lower bound on the decrement of the objective function is easily calculated before the integer solution is obtained.

In summary, the algorithm may be described as follows:

a) Select a trial by setting each variable to a desired level. The first trial is usually selected at those levels with lowest penalties.

b) Construct and update the Δb vector, i.e., $\Delta b^* = B^{-1}\Delta b$. Modify the b^* vector, i.e., $b^*_{mod} = b^* - \Delta b^*$.

c) If b^*_{mod} is feasible, obtain a greatest lower bound GLB on the objective function: $GLB = \max\{z^*_{trial}, GLB\}$; store the solution associated with GLB and go to step d). Otherwise, apply dual simplex rules.

If a dual unbound (primal infeasible) solution is found, go to step e). If the objective function obtained in any iteration is lower than the greatest lower bound obtained so far, go to step e).

d) Select a new trial. Usually this trial will have the least minimum decrement in the objective function of the possible lattice points not selected yet, i.e., the minimum of: the sum of true penalties or the largest pseudo penalties. If the least minimum decrement is greater than $z^* - GLB$, discard this solution and go to e). Otherwise, go to b).

e) If all possible integer combinations have been investigated, go to f). Otherwise, go to d).

f) The optimal mixed integer solution is the last feasible solution (if any) found with functional equal to GLB.

Since the greatest lower bound does not exist initially, the author suggests starting out with an artificial $GLB = .90z^*$. If no integer solution is found with functional in this interval, a new artificial bound is set, $GLB = .80z^*$, etc. Computational experience has shown that this criterion is good because in many cases the mixed-integer problem has a value of the optimal solution near to the value of the objective function in the optimal solution of the modified problem (4.7).

Some differences between Driebeek's (DR) and Land-Doig's (L-D) algorithm are worth noting:

1. L-D specifies only one level in one variable in each trial, while DR specifies a complete solution (a lattice point) in each trial.

2. L-D uses as a criterion for choosing the variable to be set as integer that which is furthest from an integer value. On the other hand, DR uses cost information; in each trial we know the minimum price we have to pay for setting the integer solution.

3. If a feasible solution has been found during DR, we could stop and have a measure of how far we could be from the optimal solution. In L-D this is not possible because as soon as we get the feasible solution, it is optimal.

4. In DR all equations to be used to get integer values are added in the original problem. In L-D these restrictions are added as they are needed.

In the Zionts modifications, the penalties are calculated as in Driebeek's algorithm. Once a lattice point (trial) is selected (say $x_k = j_k$, for all $k \varepsilon J$ and where j_k is an integer solution selected for x_k), the corresponding slack variables s_{kj_k} are forced to be basic variables, and the other slack variables must be nonbasic regardless of their opportunity costs.

4.6 Rebelin's Algorithm

Although using the same basic ideas presented in the above algorithm, a more simplified approach was reported by Rebelin [27].

The main features of this algorithm include:

a) It is unnecessary to introduce new 0-1 variables and constraints, thus simplifying the handling of the simplex tableau.

b) There is no difference between true and pseudo penalties.

c) Discrete variables can be handled. That is, x_k may take one of the values 0, .25, .333, which are not integers.

Assume again a maximization problem. An upper bound on the functional of the original problem (4.1) is the value of the optimal objective function (z^*) of the continuous problem (4.7). This solution could be called the "superoptimal" solution of the original problem. As the integer, or in general the discrete, variables are activated, the value of the corresponding objective function will be no greater than z^*. As soon as a feasible mixed-integer solution is found, a lower bound on the value of the objective function of problem (4.1) is obtained. If some feasible solutions are on hand, the greatest value of the objective function is selected as the lower bound. The optimal value of the functional of the mixed-integer problem will be between the upper and lower bounds.

The terms to be used in this algorithm are defined as follows:

$J = \{j \mid x_j$ is an integer (or discrete) variable$\}$.

$C = \{j \mid x_j$ is a continuous variable$\}$.

$M = \{j \mid x_j$ is in the basis$\}$.

$M^* = \{j | x_j$ is in the optimal (continuous) basis$\}$.

n = total number of variables in the original problem

$J \cup C = \{1,2,\ldots,n\}$.

A = the original matrix of coefficients (a_{ij}). $A = (A_1 | A_2)$ in problem (4.7).

x_j = the variables, $j = 1,\ldots,n$.

x_k^h = the hth value that the variable k, $k \varepsilon J$, can take;

$x_k = x_k^1,\ldots,x_k^h,\ldots,x_k^4$.

a_j = the jth column of the A matrix.

B^{-1} = the inverse matrix associated with the optimal basis.

b = the right-hand side vector in the original problem.

Z = the value of the objective function in the original problem.

β = the modified right-hand side vector in the optimal continuous table.

β_0 = the value of the objective function of the problem with modified right-hand side vector.

$*$ = a superscript indicating that the vector or matrix corresponds to the optimal continuous solution.

d_j = reduced cost of variable x_j, i.e., $z_j - c_j$. In the optimal table, $d_j^* \geq 0$.

p_k^h = penalty associated with the hth level of variable x_k, $k \varepsilon J$.

Penalty Calculations

Although Rebelin explains this calculation in a different manner, in this work the following approach will be used.

Consider the situation when the variable x_k, $k \varepsilon J$, is to be fixed at a given level x_k^h after having the optimal continuous solution. Now, x_k is either basic, $k \varepsilon M^*$, or nonbasic, $k \notin M^*$. Both cases are presented below:

Case 1

x_k Basic ($k \varepsilon J$, $k \varepsilon M^*$). In order to set x_k^h, one can subtract $x_k^h ak$ from b in the original problem (4.7), thus giving the modified b-vector as:

$$\beta = B^{-1}(b - x_k^h a_k) = b^* - x_k^h a_k^*,$$

where a_k^* is a unit vector whose non-zero element is at row R.

This procedure gives a nonbasic solution because the unit vector of the R-row is not present. A new vector to enter into the basis must be found (if it exists) which gives the minimum decrease in Z^*.

This is done with the dual simplex algorithm. If $\beta_R < 0(\beta_R \geq 0)$, one looks for that column K such that $d_K^*/a_{RK}^* = \max_{j \varepsilon C}\{d_j^*/a_{Rj}^*;\ a_{Rj}^* < 0\}$ $(=\min\{d_j^*/a_{Rj}^* > 0\})$.

The variable x_k, $K \varepsilon C$, is entered into the basis, reducing the objective function in $\beta_R(d_K^*/a_{RK}^*)$. This term is the penalty p_k^h associated with x_k^h.

If no K column is found, then no feasible solution exists when $x_k = x_k^h$ is set to $+\infty$.

Note that $\beta_R = b_R^* - x_R^h$ and $\beta_1 = b_i^*, \ldots, M$, $i \neq R$.

Note also that p_k^h is the minimum reduction in the β_0 because the dual simplex iteration to bring x_K into the basis may create new infeasibilities, ($\beta_i < 0$, $i=1,\ldots,m$, $i \neq R$).

Case 2

x_k Nonbasic ($k \varepsilon J$, $k \varepsilon M^*$). Similarly to the above argument, when $x_k = x_k^h$, the modified right-hand side becomes $\beta = b^* - x_k^h a_k^*$. The activation of x_k at its hth level has decreased Z^* by $x_R^h d_R^*$. This is the penalty associated with $x_k = x_k^h$. Again, this is the minimum reduction because some of the β_i may be negative and breaking these infeasibilities by the dual simplex will reduce β_o even more.

The algorithm for a maximization problem may be described in the following steps. A flow diagram is shown in Figure 4.2.

a) Obtain the optimal continuous solution of problem (4.7). If all x_k, $k \varepsilon J$, are integers, this is the optimal mixed-integer solution. If not, then z^* gives an upper bound on the functional of the original problem. Set an artificial lower bound $\gamma = .90z^*$.

b) Calculate the penalties p_k^h for each level of the variable x_k, $k \varepsilon J$. Set each integer variable equal to its level with minimum penalty.

c) Obtain the modified right-hand side corresponding to this lattice point. $\beta = B^{-1}(b - \sum_{k \varepsilon J} x_k^h a_k) = b^* - \sum_{k \varepsilon J} x_k^h a_k^*$.

d) If some x_k are in the optimal basis in rows $R_1, R_2, \ldots$, introduce new vectors x_j, $j \varepsilon C$ in the basis, so that $M = \{j | j \varepsilon C\}$.

e) If some $\beta_i < 0$, apply dual-simplex algorithm. If an unbound dual solution (primal unfeasible) is found, go to step f). If the value

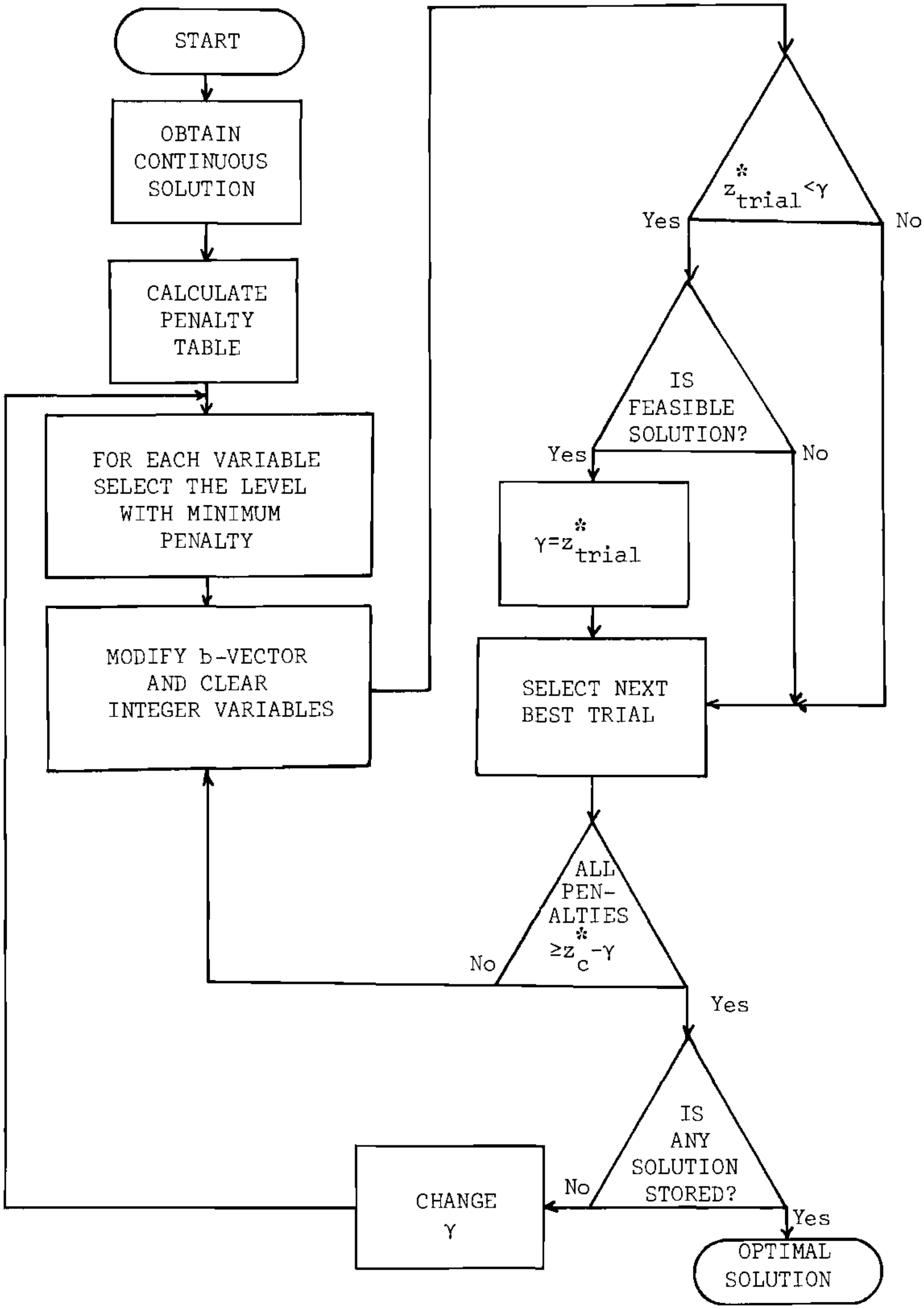

Figure 4.2 Flow Diagram of Rebelen's Algorithm for a Maximization Problem

of the objective function of the modified problem in a dual-simplex iteration falls below the greatest lower bound γ go to step f). If a feasible solution is found with $\beta_0^* > \gamma$, then the new lower bound is $\gamma = \beta_o^*$; go to step f).

f) Select another trial, i.e., a level for each x_k, $k\varepsilon J$. If the maximum penalty is greater than $z^* - \gamma$, then go to step g). Otherwise go to step c).

g) If all possible points have been selected, go to step h). Otherwise, go to step f).

h) If a feasible mixed-integer solution has been found, its optimal value is equal to γ. Otherwise, reset the value of the lower bound to $\gamma - .10z^*$. Set each integer variable to its level with minimum penalty and go to step c).

4.7 Calculation of Penalties from a Linear Programming Computer System

In this section it is shown how to obtain the penalties p_k^h associated with x_k^h from a regular linear programming computer system. In particular, the author has been working with two systems, the Burroughs ALPS and the Univac-1108's L.P.S. (Linear Programming System). Such systems have a dual output and a cost-range of the basic variables.

Penalties for Nonbasic Integer Variables

If x_k is not in the optimal basis, then the "dual output" gives its opportunity cost (reduced cost or shadow price) d_k^*; the penalties for each level are easily calculated by: $p_k^h = d_k^*(x_k^h)$.

Penalties for Basic Integer Variables

The penalties for basic variables in the optimal solution of problem (4.7) are calculated through the use of the subroutine which finds the range of the basic cost c_i. Recalling the sensitivity analysis on the basic costs (c_i, $i \varepsilon M^*$), we want to know how much these costs may be increased or decreased without changing the basis. Thus, if the new cost is $c_i' = c_i + \Delta_i$, ($c_i' = c_i + \Delta_i'$), for $\Delta_i > 0$, ($\Delta_i' < 0$), the problem is to find the greatest value of Δ_i (smallest value of Δ_i') such that the optimal basis does not change.

Since the criterion for optimality is $d_j^* > 0$, $j \notin M^*$, then as long as all d_j^* remain non-negative, no change in the basis will be made. A change of Δ_i in the basic cost c_i will change d_j^* by an amount equal to $(a_{ij}^*)\Delta_i$. If $a_{ij}^* > 0$, then d_j^* (new) > 0; but if $a_{ij}^* < 0$, then d_j^* (new) ≤ 0 when $-(a_{ij}^*)\Delta_i \geq d_j^*$. The greatest cost *increase* of c_i, Δ_i, without changing basis, is: $\Delta_i = \min_j\{-d_j^*/a_{ij}^*;\ a_{ij}^* < 0\}$.

Similarly, a change of Δ_i' in the basic cost will change d_j^* by an amount $(\Delta_i')a_{ij}^*$. Thus, if $a_{ij}^* \leq 0$, then d_j^* (new) > 0 because $\Delta_i' < 0$; but if $a_{ij}^* > 0$, then d_j^* (new) ≤ 0 when $-(a_{ij}^*)\ _i' \geq d_j^*$. The greatest cost decrease of c_i, Δ_i' is $\Delta_i' = \max_j\{-d_j^*/a_{ij}^*;\ a_{ij}^* > 0\}$.

The computer output gives both values either as Δ_i and Δ_i' (in the Univac-108) or as $(c_i + \Delta_i)$ and $(c_i + \Delta_i')$ in the Burroughs 5500. From this output, one obtains the greatest $(d_j^*/a_{ij}^*) \leq 0$, that is $-\Delta_i$, and the smallest $(d_j^*/a_{ij}^*) \geq 0$, that is $-\Delta_i'$. To calculate the penalties of a given level, say x_i^h, $i \varepsilon M^*$, one obtains the difference $(x_i^* - x_i^h)$, where x_i^* is the table.

If the difference is positive, then $p_i^h = (x_i^* - x_i^h)(-\Delta_i')$, since Δ_i' contains in its denominator a positive a_{ij}^*. If the difference is negative, then $p_i^h = (x_i^* - x_i^h)(-\Delta_i)$, since Δ_i contains a negative denominator. Of course, if the difference is zero, the penalty is also zero.

For example, in the problem reported by Land and Doig [23], the values of the integer variables in the optimal (continuous) table are $x_1 = 1.46$, $x_2 = .0099$, $x_3 = 5.0$, while the allowed values for each of the variables are $x_1^h = 0,1,2$; $x_2^h = 0,1,2$; and $x_3^h = 0,1,2,3,4,5$.

From the "primal range" output of the Univac-1108 Linear Programming System, the following information is obtained:

Limits of Range

Basic Variables (x_i)	Cost (c_i)	Label	Increment	Label	Increment
X1	77.9	Y2	-44.15	Y4	16.41
X2	76.8	Y4	-16.09	R5	341.39
X3	89.6	R5	-160.71		9999*
Y1	97.1	Y4	-99.69		9999*

*9999 is equivalent to infinity.

The first and second columns give the names and the costs of the basic variables, respectively. In this example, all integer variables are basic in the optimal tableau. The third column shows the variable to enter the basis if the original cost (c_i) is reduced more than the value of the fourth column ("increment"). The last two columns give the variable to enter into the basis if the cost (c_i) is increased by a larger amount than the last column.

The penalties associated with x_1^h are:

$$p_1^o = (1.46 - 0)(44.15) = 65.8$$

$$p_1^1 = (1.46 - 1)(44.15) = 21.67$$

$$p_1^2 = (1.46 - 2)(-16.41) = 8.36$$

Similarly for other variables, which results in the following penalty table:

h =	0	1	2	3	4	5
p_1^h	65.8	21.67	8.36			
p_2^h	.16	338	679.4			
p_3^h	803.6	642.8	482.1	321.4	160.7	0

4.8 How to Obtain the Optimal Mixed-Integer Solution

The table of penalties obtained in the last section is used to select the most promising lattice points. For each variable, the associated penalties are ranked in decreasing order. For the Land and Doig example, the arranged penalties are presented below, where the number between parentheses indicates the level of the variable associated with the penalty. The new array is:

Variable	Penalties (Level)			
x_1	8.36 (2)	21.67 (1)	65.8 (0)	
x_2	.16 (0)	338 (1)	679.4 (2)	
x_3	0 (5)	160.7 (4)	321.4 (3)	482.1 (2)
	642.8 (1)	803.6 (0)		

The first trial is made by setting each integer variable equal to that level which has the lowest penalty. In the example above, to obtain the solution of the first trial the following equations are added to the original set of constraints: $x_1 = 2$, $x_2 = 0$, and $x_3 = 5$. The second trial is made by setting $x_1 = 1$, $x_2 = 0$, and $x_3 = 5$, etc.

Note that the addition of these constraints does not follow Rebelin's algorithm, which, instead of adding new equations, provides only for changes in the b-vector. However, when using a linear programming computer code, it may be easier to add the corresponding integer constraints rather than modify the code itself as Rebelin suggested, the reason being that in using a branch and bound approach, we generally are limited to a few integer variables.

The author has written a FORTRAN program which follows closely Rebelin's algorithm. However, this program is recommended only for small problems, because it does not use special optimization subroutines as the Univac-1108 or Burroughs linear programming codes do. The program is shown in Appendix B.

CHAPTER V

AN EXAMPLE OF THE APPLICATION OF MODEL B

5.1 Introduction

This chapter presents an application of Model B which was described in Chapter IV. The problem was suggested by the American Can Company, but it may appear in other manufacturing systems. The problem situation is as follows:

(a) The manufacturing firm has available customer demand forecasts for each product over a specified planning period.

(b) Present production facilities consist of a complex of complete plants and satellite plants with known production capabilities and production costs.

(c) Capital expansion projects to increase (or decrease) total production capacity of the system have been screened and a set of economically possible alternatives with associated costs is available.

Questions which must be answered are:

1) Which alternatives should be undertaken in order to maximize total costs (production, distribution and investment costs) over the specified planning period.

2) Which are the bottlenecks of the system.

3) What is the change of profit by relaxing that bottleneck.

4) What is the effect on the system of building a new plant or closing an existing plant.

The study of individual plants would undoubtedly lead to sub-optimization and excessive investment. On the other hand, if all plants and alternatives for expansion are considered together, then an "optimal" plan for expansion may be obtained. It cannot, of course, be claimed that the plan is a true optimum, since only a finite set of alternatives is included in the model.

The relationships between plants and customers, and between plants themselves, may be expressed as linear inequalities, such as those in Model B. If a situation does not include item (c) above, then a regular production allocation model may be applied. This situation implies that there is sufficient capacity in the system and more important, that the system's structure does not change. However, sooner or later the system has to change its structure in order to follow the changes in the environment, namely, changes in demand, new products, increase in cost in some plants, changes in technology, etc.

The set of available alternatives, as stated in point (c) above, will permit selection of the best alternatives in order to meet the new situation. These alternatives are included in the model by means of integer variables. The complete set of variables and constraints, then, fits a mixed integer model, such as Model B presented in Chapter III.

In other words, with a regular linear programming model, management may obtain the best configuration of the system, assuming that its structure does not change. This does not mean that the variables included in the model do not change from period to period, but it does mean that the number of plants and their capacity cannot be changed. A mixed-integer programming model allows the inclusion of alternatives, such as increasing or reducing the number of plants, centralizing or decentralizing the system, and buying or renting equipment. In this manner, management can see the effect of some alternatives before making the final decision.

In this chapter it will be shown how the model was constructed and how the alternatives may be selected.

5.2 The System

An initial decision was made to select a typical product and a representative system of the firm. Can manufacturing was selected as the system to be studied.

The production process for can making consists of the operations shown in Figure 5.1. These operations are: 1) coil cutting, 2) coating and lithographing, 3) end presses, 4) assembly lines, and 5) packaging.

Following the decision that can manufacturing was the most appropriate one for application of the expansion planning model, the next step was to select the system.

The criteria used were 1) that the system should be sufficiently large to represent a situation typical of other systems in the firm, 2) that the system's boundaries be easily defined, and 3) that the data to be fed into the model be available.

The can-making complex is made up of seven plants. Some of them are able to take the raw material received from the mills and perform every step necessary to process it into the final products.

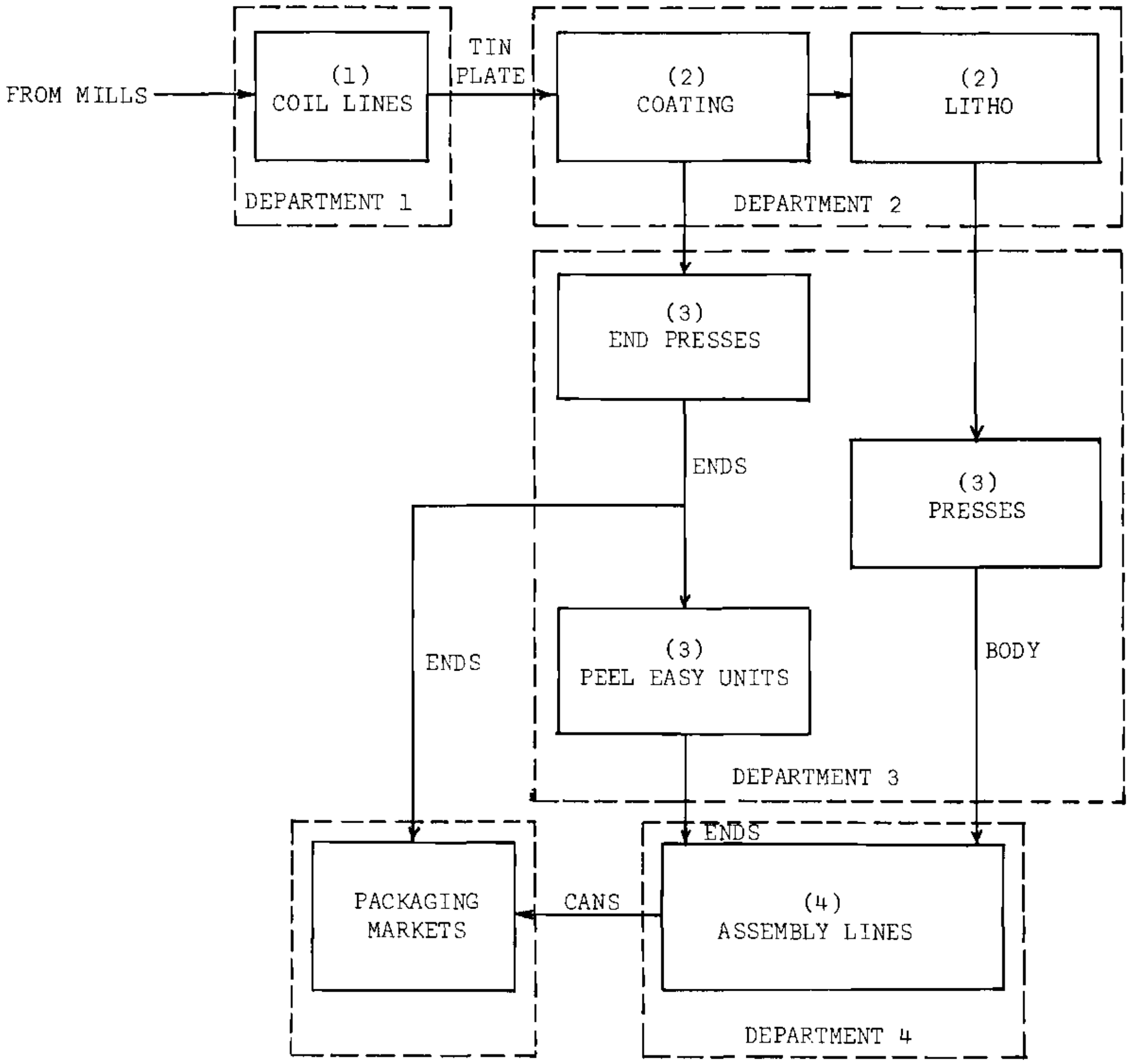

Figure 5.1 The Can-Making Process

These are called complete plants. Other plants have only assembly-line facilities; they are called satellite plants.

This complex of plants may be considered as a closed system. That is, with only few exceptions, no input from other subsystems or output to other subsystems of the firm was allowed. However, for those systems which could receive from or ship to other manufacturing subsystems, additional variables and constraints were added to the model, as in the case reported by Kendrick [15].

From records of past sales, a table was made in which each row represents one product and each column a market. In one column (market) there may be more than one customer, but for the purposes of this study the identification of each customer is lost.

The number of products was more than 30. It was found advisable to group together some of the products which had few differences among them into a representative group. Two criteria were used in grouping the products:

1. The cans should have approximately the same size.

2. The group should be manufacturing-homogeneous; that is, all cans in one group must be able to be produced by the same type of machinery.

The above procedure resulted in six beer and beverage can groups and six sanitary can groups. These groups, which from now on will be called a "representative groups" or simply a "product," are labeled B1, B2, B3, B4, B5, B6, and S1, S2, S3, S4, S5, S6, respectively.

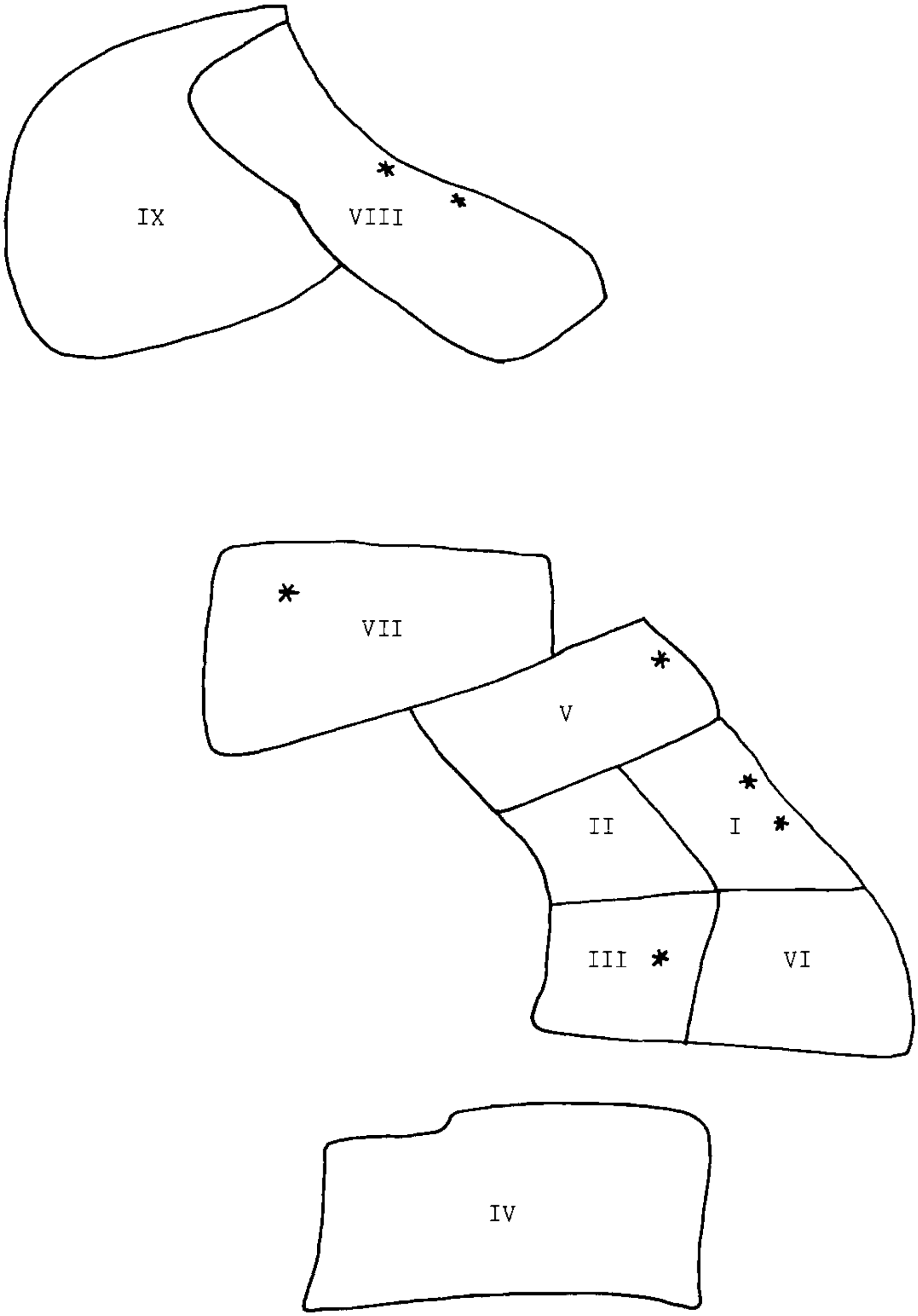

Figure 5.2 Location of Plants and Zoning at Markets

The aggregate demand for one representative group in one location is then the sum of demands for all can-size members of this group in that location.

It was observed that most of the market locations are clustered around the plants and around certain towns. Therefore, to reduce even more the dimensionality of the problem, nine regions or market zones were defined. The criteria for drawing the boundaries for each region were as follows:

1. The nearness of a plant. One region was assigned to each plant, or a group of them if they were nearby, on the basis that the furthest point in the region be no more than 50 miles (approximately) from the plant.

2. The concentration of customers. The location of each market in a region is approximately at the same distance from all the supplier plants.

After this classification, the number of regions was reduced from more than 60 to only 9.

The locations of the plants and the zones are shown in Figure 5.2. The plants are indicated by an "*".

Processes in a Plant

In this study, only the assembly-line departments were included in the model. Within each department, several sections or types of lines have been considered. Thus, plant A has three processes, namely, line B1, line B2, and line B3.

The reasons that only the assembly lines were used in the model were as follows:

a) Data on costs, standard times, etc., were readily available.

b) This department is common to all plants, both complete and satellite.

c) Most of the investment decisions concern this department.

d) The inclusion of other departments would complicate the description of the application without giving new information.

The relationships between other departments and the assembly lines may be easily added into the model.

The unit variable costs in the assembly lines were used to estimate the total unit variable cost for each product in each plant.

Each assembly-line department has either beer and beverage (BB) lines or sanitary (SN) lines or both.

There are three types of BB lines and four types of SN lines, each one with a definite capability of processing a range of sizes of cans. For example, beer and beverage line type 1, is able to produce only small sizes of cans, while line type 2 may produce small or medium sizes of cans. Since some of the products may be processed in a number of different lines, it was necessary to identify the line in which the product is manufactured.

5.3 Definition of Variables

The linear programming computer code used in this study permits the use of at most six alphanumeric symbols to define a variable. The code designation which is used to describe variables in the models is in the form XX YY ZZ. The first two terms (XX) define the representative group of products. The second two terms (YY) define the particular plant and the production line where the product is manufactured, and the last two terms (ZZ) define the time period of manufacture and the market region for which the product is produced.

In order to identify a variable easily, the following mnemonic names were assigned to each group of the symbolic code:

XX:

B1 - beer and beverage representative product 1

...
...

B6 - beer and beverage representative product 6

S1 - sanitary representative product 1

...
...

S6 - sanitary representative product 6

As stated previously, each of these representative groups is well defined in terms of the can sizes.

YY:

The first Y indicates the plant and the second Y the type of line where the product XX is manufactured. Since the beer and beverage cans cannot be produced in sanitary lines, and vice versa, the symbol XX indicates implicitly whether the line is BB or SN.

The symbols used to identify each plant are as follows: A, F, G, H, K, L, and V. Thus, for example,

B1 F1 identifies the beer and beverage representative product 1, produced in plant F, BB line type 1.

S4 G3 identifies the sanitary can representative product 4, produced in plant G, SN line type 3.

B6 A2 identifies the beer and beverage representative produced 6, produced in plant A, BB line type 2.

ZZ:

The first Z gives the period in which the variable is in operation and the second Z gives the number of the zone where the product XX is sent. The first Z takes the values of 1, 2, 3. The second Z may take any integer from 1 to 8. For example,

B1 F1 18 means BB representative product 1 produced in plant F, BB line type 1, during the first period, and sent to zone 8.

S4 G3 Z3 means SN representative product 4, produced at plant G, in SN line type 3, during the second period, and sent to zone 3.

5.4 Constraints

Only two groups of constraints were considered in this example:

The capacity constraints of each line and the market constraints in each region and for each product (of course, the integer and the

S5 means sanitary assembly line type 5.

WW indicates the period of time when the constraint is in operation. For example, T2 means second period.

Some examples of the complete labels for the capacity constraints follow:

PF B1 T1 identifies the row for capacity of plant PF, beer and beverage line type 1, during the first period of the planning horizon.

OK S2 T2 identifies the row for capacity of plant OK, sanitary line type 2, during the second period of the planning horizon.

In the market constraints, UU represents the name of the product. The first U indicates whether it is beer and beverage (B) or sanitary can (S). The second U gives the type of product, i.e., 1, 2, 3, 4, 5, or 6. For example, S3 is sanitary can type 3.

The VV indicates the zone number. The first V is always Z (zone), the second is a number from 1 to 9 which defies the zone. For example, Z8 indicates the demand in zone 8.

Finally, the WW in the markets constraints represents the time period when the market constraint is in operation. Thus, T2 means second period.

Some examples of the complete labels for the markets constraints follow:

S3 Z5 T1 identifies the row constraint for the demand for product S3, in zone 5, during the second period.

B4 Z1 T3 identifies the row constraint for the demand for the beer and beverage can type 4, in zone 1, during the third period.

5.5 Data Collection

Table 5.1 shows the capabilities and capacities of each line in each plant and the demand for each product in each zone. Each row is one product. The columns are divided into two sections--production and demand. In the production section, each type of line in every plant is indicated by one column. In the demand section, one column represents one region or market.

A number in the production section is the standard time (in hours per unit of cans*) needed to process a product (row) in some line at one plant (column). The absence of a number in a cell means that this product (row) cannot be produced in that assembly line of a plant (column). For example, product B1 may be produced in plant A (lines B1 and B2) and in Plant F (lines B1 and B2), and has demand in zones 1, 2, and 8.

The last row in the production section gives the capacity of each line in each plant (hours per year). If one plant has several identifical assembly lines, the capacity was added for all of them and considered as a single line.

For those plants which have seasonal demands, the annual capacity was reduced accordingly. If this is not done, then the solution of the model may indicate that one of these plants should produce out of season. Since no inventory is allowed in this model, the reduction of plant capacity is used to limit the production of some plants

---

*A unit of cans is arbitrarily defined and is confidential information of the American Can Company. For this paper, the knowledge of this number is not important.

Table 5.1 Capacities and Capabilities of Each Plant and Demand for Each Product and Each Zone

	PRODUCTION SECTION																		DEMAND SECTION								
	A=LA			F=PF			G=SG		H=HR			K=OK			L=LN		V=VY									
LINES	B1	B2	B3	B1	B2	B3	S1	S3	S1	S2	S5	S1	S2	S3	S	S1	S1	S3	1	2	3	4	5	6	7	8	9
PRODUCT B1	40.6	40.6		41.7	41.7														93	16						90	
B2	41.2	41.2																								25.6	
B3			64.5			52.1													27							7.2	
B4			64.5			52.1													69							73	12
B5		55.5			51.3														6							8	30
B6					52.6														45								
S1							47.6		44.4			49.6				45.4	45.4		18.5	9	29	6.5	2		8	20	
S2									48.1	48.1		49.1				45.1	36.8		23	13	18			2	2	151.6	
S3							52.6		45.3	45.3		52.6	52.6			44.2	39		23.2	25.3	16.5	11.5	20.5	14.8	40	30.9	5.2
S4								77.0		75			83.3	83.3				77	5.1	4.2	8	2.6		1.2		26	
S5								122			156			122	159			159	24	8.2	4.5		.5	.7	2.4	4.2	
S6											200				200					1			2			.8	
CAPACITY LINE	16	8	4	12	8	8	2.5	2.5	20	2.5	2.5	20	2.5	2.5	2.5	2.5	2.5	2.5									

to the months or weeks when the demand appears.

A number in one cell in the demand section indicates the annual demand for a product (row) in a market (column).

With this tabulation, one has all information needed in the matrix of coefficients in the model. Thus, if one reads across a row, the cells with numbers indicate the locations where the product can be manufactured. If one reads along a column, one can determine the products which a given line can produce, and the last cell gives the capacity of the line.

For each time period, there are 18 constraints relating the capacity of each line and 52 constraints for the markets, giving a total of 70 constraints per period. The capacity constraints are constructed by reading down a column in Table 5-1. For example, take plant G. The time spent in producing S1 and S3 cans must not exceed 2500 hours per year.

The market constraints are constructed by reading a row. For each filled cell in the demand section, one market constraint is obtained. For example, row B1 in Table 5.1 generates three market constraints (Z1, Z2, and Z9), while row B2 generates only one constraint for zone 8.

According to Model B of Chapter III, the cost associated with each continuous variable is made up of two components: 1) the unit variable cost in a plant and 2) the transportation cost from the plant to a market.

The total unit variable cost of each can in each plant was estimated with the unit variable cost in the assembly lines. It was assumed that the unit variable cost from the beginning of the process to a point before the assembly lines are reached is proportional to the variable cost per can in the assembly department.

This assumption may be checked with real data later on. However, for this study it was necessary to assume some costs in order to bring the total variable cost to approximately its real figure. Otherwise, the investment cost included in the model would be disproportionally large, resulting in a biased solution of the model.

The transportation cost is made up of two costs--the fixed cost for loading and unloading (dollars per trip) and the variable cost (dollars per hour). In order to calculate the delivery cost from one plant to a market, a speed of 30 miles per hour was assumed. Thus, the cost for a trip is given by

$$C_t = f + (v)(d)/30$$

where C_t is the cost of trip (dollars per trip), f is the fixed cost, v is the variable cost (dollars per hour), d is the average distance between a plant and the customers in a zone, and 30 is the average speed.

In order to obtain the transportation cost per unit of cans (see note on page 128), the cost per trip C_t was divided by the average units of cans that can be transported in a trip; thus, the cost of delivery per unit of cans is obtained by

$$C_D = C_t/U_t$$

where C_D is the cost of delivery (dollars per unit of cans); C_t is the cost of a trip (dollars per trip); and U_t is the number of cans that can be transported in one trip (units of can per trip).

The total unit cost of a product f.o.b. in the market is $C_D + C_v$, where C_v is the total unit variable cost of production.

Before describing the computer runs, the main points of the model described above may be summarized as follows:

In each period the model has 70 constraints and 135 variables. There are 18 constraints relating to the capacity of each assembly line in each plant and 58 market constraints which define the demand for each product in each zone and in each period of time.

Each one of the 135 variables included in each period of time specifies the product, where it is manufactured (the line and the plant), the time period and the zone number where it is shipped.

Before describing the computer runs, some characteristics of the model used to describe the example explained above should be pointed out.

The model used in this application corresponds to Model B of Chapter III with the following differences:

a) The objective in this example is to minimize the total cost of production during three periods.

b) Since only the assembly departments were included in the example, intershipments of semifinished products were not considered.

c) The raw material and capital constraints were not included in the example.

The planning is made for three periods. In each period the model has 70 constraints and 135 variables.

There are 18 constraints relating to the capacity of each assembly line in each plant and 58 market constraints which define the demand for each product in each zone and in each period of time. Each one of the 135 variables included in each period of time specifies the product, where it is manufactured (the line and plant), the time period, and the zone number to which it is shipped.

5.6 Computer Runs

In order to show the applicability of the model and its method of solution, the following hypothetical situation was designed.

It is assumed that the planning horizon is made up of three periods of time. The expected increase in the demand for each product in each period is shown on the following page.

Product	Period 1	Period 2	Period 3
B1	1.0	1.0	1.0
B2	1.0	1.2	1.4
B3	1.1	1.2	1.25
B4	1.1	1.2	1.3
B5	1.0	0.9	0.8
B6	0.8	1.0	1.4
S1	1.02	1.04	1.06
S2	1.1	1.01	1.02
S3	1.05	1.1	1.2
S4	1.1	1.2	1.3
S5	1.0	1.01	1.01
S6	1.0	1.0	0.8
Discount Factor	1.0	0.8	0.7

For example, the increase in demand for B4 in the third period is 140 per cent more than the actual demand.

The first approach to generating the investment alternatives was to run a regular linear programming and find the infeasibilities and slacks of the system. Thus, the first run indicated that new capacity was needed in beer lines and that excess capacity existed in some of the sanitary lines. On this basis, eight investment alternatives were proposed.

A description of each alternative follows. The symbol between parentheses represents the 0-1 variable.

Alternative 1: (ISGITI). Close one line S1 and one line S3 in plant SG beginning with period 1. The saving in fixed costs during the planning horizon was estimated at 0.17 dollars.

Alternative 2: (ISG2TI). Close one line S3 in plant SG beginning with the first period. The savings were estimated as 0.05 dollars.

Alternative 3: (IPFITI). Close one beer line B1 at PF plant during the first period, with savings of 0.18 dollars.

Alternative 4: (IHRITI). Close one line S1 at HR plant beginning with the first period. Expected savings are 0.3 dollars.

Alternative 5: (ILAIT2). Open a beer line B1 at plant LA beginning with the second period, with an estimated cost of 0.5 dollars.

Alternative 6: (ILA2TI). Similar to alternative 1, but beginning with period 1. The estimated cost is 1.5 dollars.

Alternative 7: (IOKIT2). Close one S1 line during the second period and another S1 during the third period, at plant OK. The estimated savings are 0.5 dollars.

Alternative 8: (IHR2TI). Build a new S2 line at HR plant beginning with the first period; the estimated cost is 0.8 dollars.

A printed output list of all variables, continuous and integer, is shown in the Appendix.

Once the alternative investment decision variables are defined, a linear programming run using the Burroughs linear programming code was made in order to calculate the penalties associated with each level

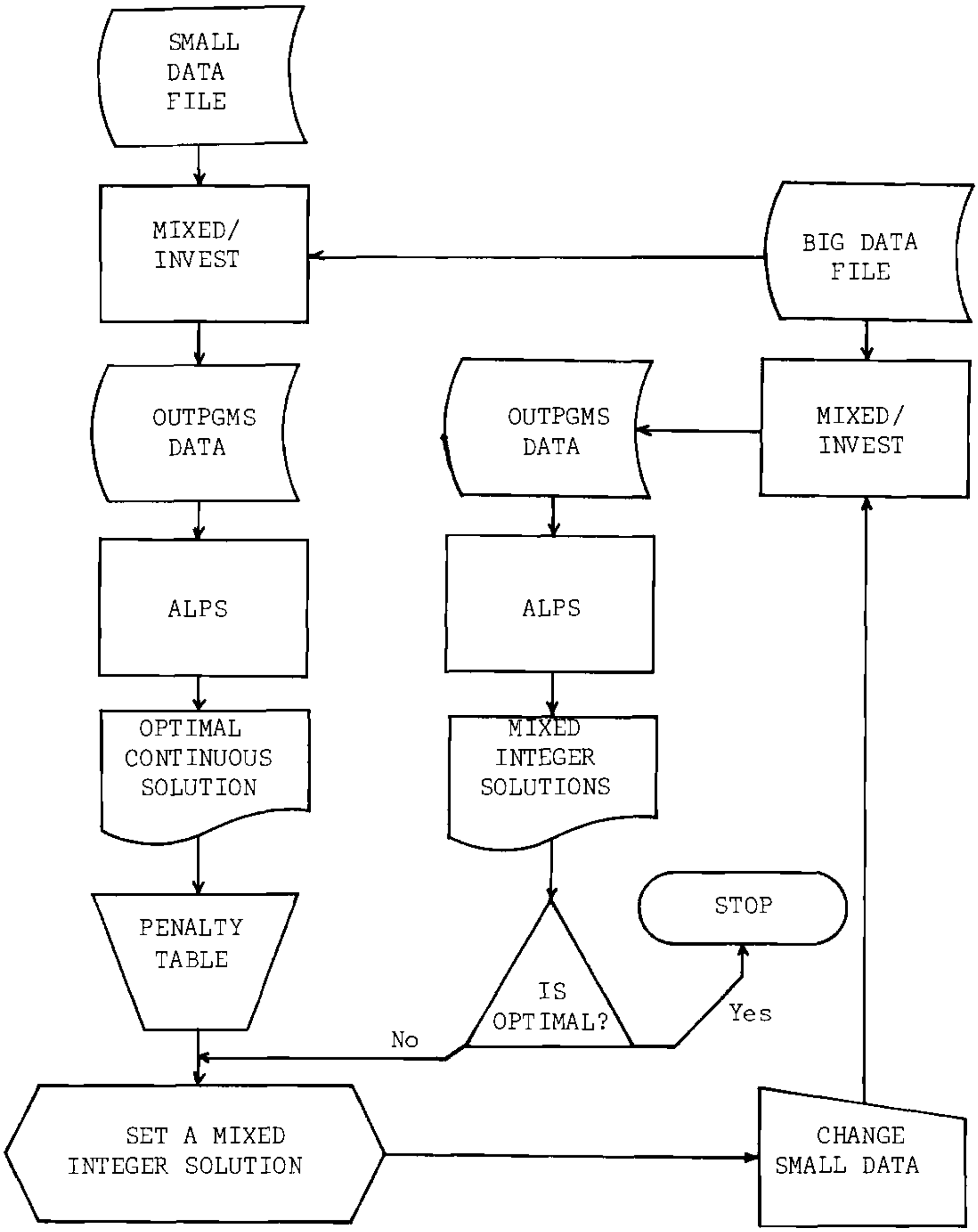

Figure 5.3 Flow Diagram of the Remote Computer Program

of the integer variables. Then, eight new constraints were added to force the integer variables to take 0 or 1 values.

The values of the b-vector corresponding to these new constraints were changed in each trial, until an optimal solution, or at least a feasible solution, was obtained.

The computer runs were made using a remote terminal connected with the B-5500 computer at the Georgia Institute of Technology Computer Center. This method of operation was selected because it permitted a more rapid change of data before starting each trial and also the basic data remained in the computer from one day to another.

Figure 5.3 shows the flow diagram of the procedure used in the remote computer operation. Two data files were used: Small File and Big File. The "small data file" contained the following information:

1. The discount factors for each period.

2. The forecast demand factors for each product and for each period.

3. Additional constraint names to be added to the "Big Data File" which were used to force the variables to integer values.

4. Additional columns (investment decision variables) and rows data to be added to the "Big Data File" and which corresponded to the investment alternatives.

5. The values of the portion of the b-vector corresponding to integer variables.

In the first run, namely, when all variables were permitted to be continuous variables, the constraints in point 3 above were of the

the type "less than," and the b-vector values of point 5 were set to 1. Thus, in this run each variable-to-be-integer was allowed to take any positive value no greater than 1. After calculating the penalties, the type of these constraints was changed to "equal to."

The "Big Data File" consisted of the data matrix corresponding to all continuous variables. These data were set as indicated in the ALPS (Burroughs) manual. The information contained in this Big Data File may be divided into three groups.

1. General information (size of the matrix, output formats, name of the program, etc.).

2. Row identification. Name and type of each constraint. The types specifies if a constraint is "less than or equal to," "equal to," or "greater than or equal to."

3. Matrix data. For each column, the rows with non-zero values are included with their respective values.

In the Big Data File, a cost row was identified for each period; thus, COST1, COST2 and COST3 indicated the cost rows for period 1, 2, and 3, respective (see Appendix A-2). The elements in these cost row vectors had to be modified by the corresponding discount factor. Similarly, the demand for each product and each period had to be modified by the forecast demand factor given in the Small Data File (see Appendix A-1).

Both the Small Data File and Big Data File were combined in an ALGOL program, called Mixed/Invest, to produce a new matrix which contained all modifications and additions for a given run. The output of

Mixed/Invest, called "OUTPGMS Data," was the input for ALPS. Appendix A-3 contains OUTPGMS data and Appendix B-1 contains the Mixed/Invest ALGOL programs.

In the first run, the integer variables were allowed to take any positive value no greater than 1. Penalties were then obtained from the optimal continuous table, and the first mixed-integer trial was set by assigning to each variable the level with minimum penalty. In the table below, the following information pertaining to the example is shown.

a) The optimal continuous values for the investment decision variables corresponding to the example.

b) The penalties associated with each level.

c) The value of each integer variable selected in the first trial.

Variable	Value in the Optimal Table	Penalties 0	Penalties 1	Value in the First Trial
ISG1TI	.8	.118	*	0
ISG2TI	0.0	0	.5	0
IPFITI	1.0	.18	0	1
IHRITI	1.0	.3	0	1
ILAIT2	0.0	0	.44	0
ILA2TI	.43	*	1.1	1
IOKIT3	1.0	.5	0	1
I1+R2T1	.014	*	.78	1

* = infinity.

Two examples of the penalty calculation follows:

Variable: ISG1T1

Actual profit (cost) = -.17

Value in the optimal table = .8

From the sensitivity analysis output the max $((dj/a_{rj})\ 0);\ a_{Tj}<0)$ and the min $((d_j/a_{rj})\geq 0;\ a_{rj}>0)$ were obtained.

Lower range = OPEN then, max $(d_j/a_{rj}\leq 0) = -\infty$ and penalty (1) = ∞.

Upper range = -.0330 then,

min $(d_j/a_{rj}\geq 0) = 0.0330 - (-.17) = .147$

and penalty (0) = (.147)(.8-0) = .1176.

Now, for IPF1T1, its actual profit or (-cost) is -.18 and its value in the optimal table is 1. Then upper range is 0; then penalty (0) = (1-0)(0-(-.18)) = .18.

It should be noted that the values for some variables must be fixed for all trails because the penalties for some levels are equal to infinity. Using this fact, another run was made in order to obtain a better (lower) bound on the objective function. The values fixed were ISG1T1=0, ILA2T1=1, IHR2T1=1, while other variables to-be-integer were allowed to take any positive value no greater than 1.

The results of this run indicated that an optimal solution was obtained because the values for all investment variables were integers; thus, no more branching was needed.

The results of this problem were used to generate another set of 12 new alternatives, whose results were given to the American Can

Company. Appendix A-4 shows a sample of a mixed-integer solution printout.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

A. Conclusions

In the development of this work, it was shown that long-range proposals, such as investment alternatives, can be incorporated in a linear programming model, in order to plan the expansion of a firm.

The model and its solution permitted not only the evaluation of a set of investment alternatives, but also the generation of new proposals.

The solution to the expansion problem also gives the solution of the medium-range planning problem, namely, the allocation of facilities to consumer regions.

The development of new algorithms to solve mixed-integer programming problems has permitted the solving of many problems which a few years ago were discussed only theoretically.

The solution of each step of the practical problem took about ten minutes. Due to the computer system difficulties encountered, the solution in each branch was started from the initial table, instead of the optimal continuous solution. It is believed that with improved computer facilities this time could be reduced at least 90 per cent, because most of the computer time in each run was spent in drawing out artificial vectors from the solution. Also, in an improved computer system, the penalty calculation and the selection of the level of each integer variable may be incorporated in a single code so that the computer may obtain the optimal solution automatically.

The method of solution permits the addition of new investment alternatives at any time. After an initial trial is made, the decision maker can introduce a new set of alternatives, and by running again a linear programming code, the penalties for these new integer variables may be calculated.

Also it is possible to look for other non-optimal alternatives in order to trade-off for intangible factors.

B. Recommendations for Further Study

Suggestions for extension of this work follow:

1. Since some of the actual production costs are not linear, an extension may be to divide the cost into fixed-fixed, fixed-variable, and variable cost and include these terms in the model.

2. Consideration of the intangibles either as an extension of the model or as a methodology once the optimal solution is obtained might be tried.

3. In most of the firms, the profit increase is not a sufficient criterion but a minimum rate of return on investment is required. Thus, in some studies the objective function should be changed to maximize rate of return, rather than maximize profit or minimize cost.

In order to generate alternatives, a simulation program may be used. When a feasible non-optimal solution has been obtained, one can calculate the greatest improvement that may be obtained by trying other solutions. Therefore, in some cases it is not necessary to investigate all lattice points, because the greatest improvement that might be obtained is small compared with the possible effect of intangibles and error in the data collection. The restriction that the demand for the next period is known may be relaxed by assigning a probability distribution to the demand for each product. Using a Montecargo Simulation, a series of demand vectors may be generated. Then a mixed-integer solution may be obtained for each vector by using sensitivity analysis. In this way, the effect of changing demand on the investment decision can be calculated.

APPENDIX A

SMALL DATA

SML /SML LISTED AT 18:30 ON 69259 : SMSML /B1N1A09 51%

```
1,,,8,,7,
1,1,1,1,
2,1,1.2,1.4,
3,1.1,1.2,1.25,
4,1.1,1.2,1.3,
5,1,,9,,8,
6,,8,1.0,1.4,
7,1.02,1.04,1.06,
8,1.1,1.01,1.02,
9,1.05,1.1,1.2,
10,1.1,1.2,1.3,
11,1,1.01,1.01,
12,1,1,,8,
NEWRWS
      1       +RISG11
      1       +RIPF11
      1       +RIHR11
      1       +RILA11
      1       +RILA21
      1       +RIOK12
      1       +RISG22
      1       +RIHR21
      1       +RIOK21
      1       +RIPF21
      1       +RIPF32
      1       +RILA31
ENDRWS
NEWCUL
        ISG1T1CST    -     .17
        ISG1T1SGS1T1 2500.
        ISG1T1SGS3T1 2500.
        ISG1T1SGS1T2 2500.
        ISG1T1SGS3T2 2500.
        ISG1T1SGS1T3 2500.
        ISG1T1SGS3T3 2500.
        ISG1T1RISG11    1.
        ISG2T2CST          .5
        ISG2T2SGS1T2-2500.
        ISG2T2SGS1T3-2500.
        ISG2T2RISG22    1.
        IPF1T1CST    -     .25
        IPF1T1PFB1T1 8000.0
        IPF1T1PFB1T2 8000.
        IPF1T1PFB1T3 8000.
        IPF1T1RIPF11    1.
        IHR1T1CST    -     .5
        IHR1T1HRS1T1 5000.
        IHR1T1HRS1T2 7500.
        IHR1T1HRS1T3 7500.
        IHR1T1RIHR11    1.
        ILA1T1CST         0.5
        ILA1T1LAB1T1-4000.
        ILA1T1LAB1T2-4000.
        ILA1T1LAB1T3-4000.
        ILA1T1RILA11    1.
        ILA2T1CST         1.5
        ILA2T1LAB3T1-4000.
        ILA2T1LAB3T2-4000.
        ILA2T1LAB3T3-4000.
        ILA2T1RILA21    1.
```

```
        IOK1T2CST    -    1.
        IOK1T2OKS1T210000.
        IOK1T2OKS1T310000.
        IOK1T2RIOK12     1.
        IHR2T1CST        -.8
        IHR2T1HRS2T1-2500.
        IHR2T1HRS2T2-2500.
        IHR2T1HRS2T3-2500.
        IHR2T1RIHR21     1.
        IOK2T1CST    -    .75
        IOK2T1OKS1T1 5000.
        IOK2T1OKS1T2 7500.
        IOK2T1OKS1T310000.
        IOK2T1RIOK21     1.
        IPF2T1CST    -    .25
        IPF2T1PFA1T1 4000.
        IPF2T1PFA1T2 8000.
        IPF2T1PFA1T312000.
        IPF2T1RIPF21     1.
        IPF3T2CST    -    .15
        IPF3T2PFA2T2 4000.
        IPF3T2PFA2T3 4000.
        IPF3T2RIPF32     1.
        ILA3T1CST    -   1.
        ILA3T1LAA2T1 4000.
        ILA3T1LAA2T2 4000.
        ILA3T1LAA2T3 4000.
        ILA3T1RILA31     1.
ENDCOL
NEWBRW
              RISG11     1.0
              RIPF11     1.0
              RIHR11     1.
              RILA11     1.0
              RILA21     1.0
              RIOK12     1.
              RISG22     1.0
              RIHR21     1.0
              RIOK21     1.
              RIPF21     1.
              RIPF32     1.
              RILA31     1.
              RIPF21     1.
              RIPF32     1.
              RILA31     1.
ENDBRW
NOCHANGE
```

APPENDIX B

BIG DATA (A SAMPLE)

```
                                BIG     /IEUSERS     LISTED AT 18131 ON 69259 + BIG     /BIN1A09

MIXED    300   800
CONTROL
OUTPUT
     0     1     4     4     1
END
HEADING
MIXED INTEGER PROGRAMMING
END
READ DATA
CARD
ROW ID
     1     CST
     5     &PFB1T1    &PFB2T1    &PFB3T1    &LAB1T1    &LAB2T1
     5     &LAB3T1    -B121T1    -B122T1    -B128T1    -B228T1
     5     -B321T1     B328T1    -B421T1    -B428T1    -B429T1
     4     -B521T1    -B528T1    -B529T1    -B621T1
     5     &OKS1T1    &OKS2T1    &OKS3T1    &OKS5T1    &SGS1T1
     5     &SGS3T1    &LNS1T1    &VYS1T1    &VYS3T1    &HRS1T1
     5     &HRS2T1    &HRS5T1    -S121T1    -S122T1    -S123T1
     5     -S124T1    -S125T1    -S127T1    -S128T1    -S221T1
     5     -S222T1    -S223T1    -S226T1    -S227T1    -S228T1
     5     -S321T1    -S322T1    -S323T1    -S324T1    -S325T1
     5     -S326T1    -S327T1    -S328T1    -S329T1    -S421T1
     5     -S422T1    -S423T1    -S424T1    -S426T1    -S428T1
     5     -S521T1    -S522T1    -S523T1    -S525T1    -S526T1
     5     -S527T1    -S528T1    -S622T1    -S625T1    -S626T1
     1     -S628T1
     5     &PFB1T2    &PFB2T2    &PFB3T2    &LAB1T2    &LAB2T2
     5     &LAB3T2    -B121T2    -B122T2    -B128T2    -B228T2
     5     -B321T2     B328T2    -B421T2    -B428T2    -B429T2
     4     -B521T2    -B528T2    -B529T2    -B621T2
     5     &OKS1T2    &OKS2T2    &OKS3T2    &OKS5T2    &SGS1T2
     5     &SGS3T2    &LNS1T2    &VYS1T2    &VYS3T2    &HRS1T2
     5     &HRS2T2    &HRS5T2    -S121T2    -S122T2    -S123T2
     5     -S124T2    -S125T2    -S127T2    -S128T2    -S221T2
     5     -S222T2    -S223T2    -S226T2    -S227T2    -S228T2
     5     -S321T2    -S322T2    -S323T2    -S324T2    -S325T2
     5     -S326T2    -S327T2    -S328T2    -S329T2    -S421T2
     5     -S422T2    -S423T2    -S424T2    -S426T2    -S428T2
     5     -S521T2    -S522T2    -S523T2    -S525T2    -S526T2
     5     -S527T2    -S528T2    -S622T2    -S625T2    -S626T2
     1     -S628T2
     5     &PFB1T3    &PFB2T3    &PFB3T3    &LAB1T3    &LAB2T3
     5     &LAB3T3    -B121T3    -B122T3    -B128T3    -B228T3
     5     -B321T3     B328T3    -B421T3    -B428T3    -B429T3
     4     -B521T3    -B528T3    -B529T3    -B621T3
     5     &OKS1T3    &OKS2T3    &OKS3T3    &OKS5T3    &SGS1T3
     5     &SGS3T3    &LNS1T3    &VYS1T3    &VYS3T3    &HRS1T3
     5     &HRS2T3    &HRS5T3    -S121T3    -S122T3    -S123T3
     5     -S124T3    -S125T3    -S127T3    -S128T3    -S221T3
     5     -S222T3    -S223T3    -S226T3    -S227T3    -S228T3
     5     -S321T3    -S322T3    -S323T3    -S324T3    -S325T3
     5     -S326T3    -S327T3    -S328T3    -S329T3    -S421T3
     5     -S422T3    -S423T3    -S424T3    -S426T3    -S428T3
     5     -S521T3    -S522T3    -S523T3    -S525T3    -S526T3
     5     -S527T3    -S528T3    -S622T3    -S625T3    -S626T3
     1     -S628T3
END
CARD
MATRIX
```

```
B1A111COST1   4330                                                    00000061
B1A111LAB1T1    40.6                                                  00000062
B1A111B1Z1T1     1.                                                   00000063
B1A118COST1   2330                                                    00000064
B1A118LAB1T1    40.6                                                  00000065
B1A118B1Z8T1     1.0                                                  00000066
B1A211COST1   4330                                                    00000067
B1A211LAB2T1    40.6                                                  00000068
B1A211B1Z1T1     1.                                                   00000069
B1A218COST1   2330                                                    00000070
B1A218LAB2T1    40.6                                                  00000071
B1A218B1Z8T1     1.0                                                  00000072
B1F111COST1   2320                                                    00000073
B1F111PFB1T1    41.7                                                  00000074
B1F111B1Z1T1     1.0                                                  00000075
B1F112COST1   2770                                                    00000076
B1F112PFB1T1    41.7                                                  00000077
B1F112B1Z2T1     1.0                                                  00000078
B1F118COST1   4320                                                    00000079
B1F118PFB1T1    41.7                                                  00000080
B1F118B1Z8T1     1.                                                   00000081
B1F211COST1   2320                                                    00000082
B1F211PFB2T1    41.7                                                  00000083
B1F211B1Z1T1     1.0                                                  00000084
B1F212COST1   2770                                                    00000085
B1F212PFB2T1    41.7                                                  00000086
B1F212B1Z2T1     1.0                                                  00000087
B2A118COST1   2360                                                    00000088
B2A118LAB1T1    41.2                                                  00000089
B2A118B2Z8T1     1.0                                                  00000090
B2A218COST1   2360                                                    00000091
B2A218LAB2T1    41.2                                                  00000092
B2A218B2Z8T1     1.0                                                  00000093
B3A318COST1   3700                                                    00000094
B3A318LAB3T1    64.5                                                  00000095
B3A318B3Z8T1     1.0                                                  00000096
B3F311COST1   2900                                                    00000097
B3F311PFB3T1    52.1                                                  00000098
B3F311B3Z1T1     1.0                                                  00000099
B4A318COST1   3700                                                    00000100
B4A318LAB3T1    64.5                                                  00000101
B4A318B4Z8T1     1.0                                                  00000102
B4A319COST1   5100                                                    00000103
B4A319LAB3T1    64.5                                                  00000104
B4A319B4Z9T1     1.0                                                  00000105
B4F311COST1   2900                                                    00000106
B4F311PFB3T1    52.1                                                  00000107
B4F311B4Z1T1     1.0                                                  00000108
B5A218COST1   3180                                                    00000109
B5A218LAB2T1    55.5                                                  00000110
B5A218B5Z8T1     1.0                                                  00000111
B5A219COST1   4940                                                    00000112
B5A219LAB2T1    54.0                                                  00000113
B5A219B5Z9T1     1.0                                                  00000114
B5F211COST1   2860                                                    00000115
B5F211PFB2T1    51.3                                                  00000116
B5F211B5Z1T1     1.0                                                  00000117
B6F211COST1   2920                                                    00000118
B6F211PFB2T1    52.6                                                  00000119
B6F211B6Z1T1     1.0                                                  00000120
S1G111COST1   2620                                                    00000121
S1G111SGS1T1    47.6                                                  00000122
```

```
B1Z8T3    90.                                                                  00001425
B2Z8T3   256.5                                                                 00001426
B3Z1T3    27.                                                                  00001427
B3Z8T3     7.2                                                                 00001428
B4Z1T3    69.                                                                  00001429
B4Z8T3    73.                                                                  00001430
B4Z9T3    12.                                                                  00001431
B5Z1T3     6.                                                                  00001432
B5Z8T3     8.                                                                  00001433
B5Z9T3    30.                                                                  00001434
B6Z1T3    47.                                                                  00001435
OKS1T320000.                                                                   00001436
OKS2T3 2500.                                                                   00001437
OKS3T3 2500.                                                                   00001438
OKS5T3 2500.                                                                   00001439
SGS1T3 2500.                                                                   00001440
SGS3T3 2500.                                                                   00001441
LNS1T3 2500.                                                                   00001442
VYS1T3 2500.                                                                   00001443
VYS3T3 2500.                                                                   00001444
HRS1T320000.                                                                   00001445
HRS2T3 2500.                                                                   00001446
HRS5T3 2500.                                                                   00001447
S1Z1T3    18.5                                                                 00001448
S1Z2T3     9.                                                                  00001449
S1Z3T3    29.                                                                  00001450
S1Z4T3     6.5                                                                 00001451
S1Z5T3     2.                                                                  00001452
S1Z7T3     8.                                                                  00001453
S1Z8T3    20.                                                                  00001454
S2Z1T3    23.                                                                  00001455
S2Z2T3    13.                                                                  00001456
S2Z3T3    18.                                                                  00001457
S2Z6T3     2.                                                                  00001458
S2Z7T3     2.                                                                  00001459
S2Z8T3   157.6                                                                 00001460
S3Z1T3    23.2                                                                 00001461
S3Z2T3    25.3                                                                 00001462
S3Z3T3    16.5                                                                 00001463
S3Z4T3    11.5                                                                 00001464
S3Z5T3    20.5                                                                 00001465
S3Z6T3    14.8                                                                 00001466
S3Z7T3    40.                                                                  00001467
S3Z8T3    30.9                                                                 00001468
S3Z9T3     5.2                                                                 00001469
S4Z1T3     5.1                                                                 00001470
S4Z2T3     4.2                                                                 00001471
S4Z3T3     8.                                                                  00001472
S4Z4T3     2.6                                                                 00001473
S4Z6T3     1.2                                                                 00001474
S4Z8T3    26.                                                                  00001475
S5Z1T3    24.                                                                  00001476
S5Z2T3     8.2                                                                 00001477
S5Z3T3     4.5                                                                 00001478
S5Z5T3      .5                                                                 00001479
S5Z6T3      .7                                                                 00001480
S5Z7T3     2.4                                                                 00001481
S5Z8T3     4.2                                                                 00001482
S6Z2T3     1.                                                                  00001483
S6Z5T3     2.                                                                  00001484
S6Z6T3     1.                                                                  00001485
S6Z8T3     0.8                                                                 00001486
```

```
EOF                                                          00001487
START PHASE ONE                                              00001488
CST          8                                               00001489
COMPUTE                                                      00001490
CST          8                                               00001491
COST RANGING                                                 00001492
ISG1T1                                                       00001493
IPF1T1                                                       00001494
IHR1T1                                                       00001495
ILA1T1                                                       00001496
IOK1T2                                                       00001497
ILA2T1                                                       00001498
ISG2T2                                                       00001499
IHR2T1                                                       00001500
IOK2T1                                                       00001501
IPF2T1                                                       00001502
IPF3T2                                                       00001503
ILA3T1                                                       00001504
END                                                          00001505
CONCLUDE                                                     00001506
```

APPENDIX C

OUTPGMS DATA (A SAMPLE)

```
MIXED      300    800
CONTROL
OUTPUT
     0       1       4       4
END
HEADING
MIXED INTEGER PROGRAMMING
         1.00        0.80        0.70
         1.00        1.00        1.00        1.00
         2.00        1.00        1.20        1.40
         3.00        1.10        1.20        1.25
         4.00        1.10        1.20        1.30
         5.00        1.00        0.90        0.80
         6.00        0.80        1.00        1.40
         7.00        1.02        1.04        1.06
         8.00        1.10        1.01        1.02
         9.00        1.05        1.10        1.20
        10.00        1.10        1.20        1.30
        11.00        1.00        1.01        1.01
        12.00        1.00        1.00        0.80
END
READ DATA
CARD
ROW 1D
     1      CST
     5     &PFB1T1      &PFB2T1      &PFB3T1      &LAB1T1      &LAB2T1
     5     &LAB3T1      -B1Z1T1      -B1Z2T1      -B1Z8T1      -B2Z8T1
     5     -B3Z1T1      -B3Z8T1      -B4Z1T1      -B4Z8T1      -B4Z9T1
     4     -B5Z1T1      -B5Z8T1      -B5Z9T1      -B6Z1T1
     5     &OKS1T1      &OKS2T1      &OKS3T1      &OKS5T1      &SGS1T1
     5     &SGS3T1      &LNS1T1      &VYS1T1      &VYS3T1      &HRS1T1
     5     &HRS2T1      &HRS5T1      -S1Z1T1      -S1Z2T1      -S1Z3T1
     5     -S1Z4T1      -S1Z5T1      -S1Z7T1      -S1Z8T1      -S2Z1T1
     5     -S2Z2T1      -S2Z3T1      -S2Z6T1      -S2Z7T1      -S2Z8T1
     5     -S3Z1T1      -S3Z2T1      -S3Z3T1      -S3Z4T1      -S3Z5T1
     5     -S3Z6T1      -S3Z7T1      -S3Z8T1      -S3Z9T1      -S4Z1T1
     5     -S4Z2T1      -S4Z3T1      -S4Z4T1      -S4Z6T1      -S4Z8T1
     5     -S5Z1T1      -S5Z2T1      -S5Z3T1      -S5Z5T1      -S5Z6T1
     5     -S5Z7T1      -S5Z8T1      -S6Z2T1      -S6Z5T1      -S6Z6T1
     1     -S6Z8T1
     5     &PFB1T2      &PFB2T2      &PFB3T2      &LAB1T2      &LAB2T2
     5     &LAB3T2      -B1Z1T2      -B1Z2T2      -B1Z8T2      -B2Z8T2
     5     -B3Z1T2      -B3Z8T2      -B4Z1T2      -B4Z8T2      -B4Z9T2
     4     -B5Z1T2      -B5Z8T2      -B5Z9T2      -B6Z1T2
     5     &OKS1T2      &OKS2T2      &OKS3T2      &OKS5T2      &SGS1T2
     5     &SGS3T2      &LNS1T2      &VYS1T2      &VYS3T2      &HRS1T2
     5     &HRS2T2      &HRS5T2      -S1Z1T2      -S1Z2T2      -S1Z3T2
     5     -S1Z4T2      -S1Z5T2      -S1Z7T2      -S1Z8T2      -S2Z1T2
     5     -S2Z2T2      -S2Z3T2      -S2Z6T2      -S2Z7T2      -S2Z8T2
     5     -S3Z1T2      -S3Z2T2      -S3Z3T2      -S3Z4T2      -S3Z5T2
     5     -S3Z6T2      -S3Z7T2      -S3Z8T2      -S3Z9T2      -S4Z1T2
     5     -S4Z2T2      -S4Z3T2      -S4Z4T2      -S4Z6T2      -S4Z8T2
     5     -S5Z1T2      -S5Z2T2      -S5Z3T2      -S5Z5T2      -S5Z6T2
     5     -S5Z7T2      -S5Z8T2      -S6Z2T2      -S6Z5T2      -S6Z6T2
     1     -S6Z8T2
     5     &PFB1T3      &PFB2T3      &PFB3T3      &LAB1T3      &LAB2T3
     5     &LAB3T3      -B1Z1T3      -B1Z2T3      -B1Z8T3      -B2Z8T3
     5     -B3Z1T3      -B3Z8T3      -B4Z1T3      -B4Z8T3      -B4Z9T3
     4     -B5Z1T3      -B5Z8T3      -B5Z9T3      -B6Z1T3
     5     &OKS1T3      &OKS2T3      &OKS3T3      &OKS5T3      &SGS1T3
     5     &SGS3T3      &LNS1T3      &VYS1T3      &VYS3T3      &HRS1T3
     5     &HRS2T3      &HRS5T3      -S1Z1T3      -S1Z2T3      -S1Z3T3
```

```
        5       -S1Z4T3        -S1Z5T3        -S1Z7T3        -S1Z8T3        -S2Z1T3
        5       -S2Z2T3        -S2Z3T3        -S2Z6T3        -S2Z7T3        -S2Z8T3
        5       -S3Z1T3        -S3Z2T3        -S3Z3T3        -S3Z4T3        -S3Z5T3
        5       -S3Z6T3        -S3Z7T3        -S3Z8T3        -S3Z9T3        -S4Z1T3
        5       -S4Z2T3        -S4Z3T3        -S4Z4T3        -S4Z6T3        -S4Z8T3
        5       -S5Z1T3        -S5Z2T3        -S5Z3T3        -S5Z5T3        -S5Z6T3
        5       -S5Z7T3        -S5Z8T3        -S6Z2T3        -S6Z5T3        -S6Z6T3
        1       -S6Z8T3
        1       +RISG11
        1       +RIPF11
        1       +RIHR11
        1       +RILA12
        1       +RILA21
        1       +RIOK13
        1       +RISG21
        1       +RIHR21
END
CARD
MATRIX
          B1A111CST           0.004330
          B1A111LAB1T1       40.600000
          B1A111B1Z1T1        1.000000
          B1A118CST           0.002330
          B1A118LAB1T1       40.600000
          B1A118B1Z8T1        1.000000
          B1A211CST           0.004330
          B1A211LAB2T2       40.600000
          B1A211B1Z1T1        1.000000
          B1A218CST           0.002330
          B1A218LAB2T1       40.600000
          B1A218B1Z8T1        1.000000
          B1F111CST           0.002320
          B1F111PFB1T1       41.700000
          B1F111B1Z1T1        1.000000
          B1F112CST           0.002770
          B1F112PFB1T1       41.700000
          B1F112B1Z2T1        1.000000
          B1F118CST           0.004320
          B1F118PFB1T1       41.700000
          B1F118B1Z8T1        1.000000
          B1F211CST           0.002320
          B1F211PFB2T1       41.700000
          B1F211B1Z1T1        1.000000
          B1F212CST           0.002770
          B1F212PFB2T1       41.700000
          B1F212B1Z2T1        1.000000
          B2A118CST           0.002360
          B2A118LAB1T1       41.200000
          B2A118B2Z8T1        1.000000
          B2A218CST           0.002360
          B2A218LAB2T1       41.200000
          B2A218B2Z8T1        1.000000
          B3A318CST           0.003700
          B3A318LAB3T1       64.500000
          B3A318B3Z8T1        1.000000
          B3F311CST           0.002900
          B3F311PFB3T1       52.100000
          B3F311B3Z1T1        1.000000
          B4A318CST           0.003700
          B4A318LAB3T1       64.500000
          B4A318B4Z8T1        1.000000
          B4A319CST           0.005100
```

B4A319LAB3T1	64.500000
B4A319B4Z9T1	1.000000
B4F311CST	0.002900
B4F311PFB3T1	52.100000
B4F311B4Z1T1	1.000000
B5A218CST	0.003180
B5A218LAB2T1	55.500000
B5A218B5Z8T1	1.000000
B5A219CST	0.004940
B5A219LAB2T1	54.000000
B5A219B5Z9T1	1.000000
B5F211CST	0.002860
B5F211PFB2T1	51.300000
B5F211B5Z1T1	1.000000
B6F211CST	0.002920
B6F211PFB2T1	52.600000
B6F211B6Z1T1	1.000000
S1G111CST	0.002620
S1G111SGS1T1	47.600000
S1G111S1Z1T1	1.000000
S1G112CST	0.002880
S1G112SGS1T1	47.600000
S1G112S1Z2T1	1.000000
S1G113CST	0.003070
S1G113SGS1T1	47.600000
S1G113S1Z3T1	1.000000
S1G115CST	0.002920
S1G115SGS1T1	47.600000
S1G115S1Z5T1	1.000000
S1G117CST	0.003410
S1G117SGS1T1	47.600000
S1G117S1Z7T1	1.000000
S1H117CST	0.002910
S1H117HRS1T1	44.400000
S1H117S1Z7T1	1.000000
S1H118CST	0.002310
S1H118HRS1T1	44.400000
S1H118S1Z8T1	1.000000
S1K111CST	0.002130
S1K111OKS1T1	49.600000
S1K111S1Z1T1	1.000000
S1K1Z2CST	0.002470
S1K1Z2OKSL1	49.600000
S1K1Z2SN1Z2	1.000000
S1K112CST	0.002470
S1K112OKS1T1	49.600000
S1K112S1Z2T1	1.000000
S1K113CST	0.002620
S1K113OKS1T1	49.600000
S1K113S1Z3T1	1.000000
S1K114CST	0.002810
S1K114OKS1T1	49.600000
S1K114S1Z4T1	1.000000
S1K115CST	0.002620
S1K115OKS1T1	49.600000
S1K115S1Z5T1	1.000000
S1L115CST	0.002870
S1L115LNS1T1	45.400000
S1L115S1Z5T1	1.000000
S1L117CST	0.002310
S1L117LNS1T1	45.400000
S1L117S1Z7T1	1.000000

B471T3	89.700000
B478T3	94.900000
B479T3	15.600000
B571T3	4.800000
B578T3	6.400000
B579T3	24.000000
B671T3	65.800000
DKS1T3	20000.000000
DKS2T3	2500.000000
DKS3T3	2500.000000
DKS5T3	2500.000000
SGS1T3	3500.000000
SGS3T3	3500.000000
LNS1T3	2500.000000
VYS1T3	2500.000000
VYS3T3	2500.000000
HRS1T3	20000.000000
HRS2T3	2500.000000
HRS5T3	2500.000000
S171T3	19.610000
S172T3	9.540000
S173T3	30.740000
S174T3	6.890000
S175T3	2.120000
S177T3	8.480000
S178T3	21.200000
S271T3	23.460000
S272T3	13.260000
S273T3	18.360000
S276T3	2.040000
S277T3	2.040000
S278T3	160.752000
S371T3	27.840000
S372T3	30.360000
S373T3	19.800000
S374T3	13.800000
S375T3	24.600000
S376T3	17.760000
S377T3	48.000000
S378T3	37.080000
S379T3	6.240000
S471T3	6.630000
S472T3	5.460000
S473T3	10.400000
S474T3	3.380000
S476T3	1.560000
S478T3	33.800000
S571T3	24.240000
S572T3	8.282000
S573T3	4.545000
S575T3	0.505000
S576T3	0.707000
S577T3	2.424000
S578T3	4.242000
S672T3	0.800000
S675T3	1.600000
S676T3	0.800000
S678T3	0.640000
RISG11	1.000000
RIPF11	1.000000
RIHR11	1.000000
RILA12	1.000000

```
                  RILA21      1.000000
                  RIOK13      1.000000
                  RISG21      1.000000
                  RIHR21      1.000000
EOF
START PHASE ONE
CST          B
COMPUTE
CST          B
COST RANGING
ISG1T1
IPF1T1
IHR1T1
ILA1T2
IOK1T3
ISG2T1
ILA2T1
IHR2T1
END
CONCLUDE
```

APPENDIX D

OUTPUT ALPS (A SAMPLE)

```
DATE: SEPT 17, 1969              RUN STARTED AT:   1860 HOURS              B 5500 ALGOL OUTPUT FROM PROGRAM ALPS

MIXED    300    800                    WITH  A L P S-1 FOR B5500
                                       *****************************

                                             *****<[ INPUT  ]>******  TIMES:   PROC. =    1.1   ELAPSED =    1.3
CONTROL      CONTROL CARD

OUTPUT
      0      1      4      4      0      0      0      0
END
HEADING      CONTROL CARD

MIXED INTEGER PROGRAMMING
    1.00       0.80       0.70
    1.00       1.00       1.00       1.00
    2.00       1.00       1.20       1.40
    3.00       1.10       1.20       1.25
    4.00       1.10       1.20       1.30
    5.00       1.00       0.90       0.80
    6.00       0.80       1.00       1.40
    7.00       1.02       1.04       1.06
    8.00       1.10       1.01       1.02
    9.00       1.05       1.10       1.20
   10.00       1.10       1.20       1.30
   11.00       1.00       1.01       1.01
   12.00       1.00       1.00       0.80
READ DATA    CONTROL CARD

CARD
CARD

THE PROBLEM HAS  224 ROWS AND  634 COLUMNS, 1491 MATRIX ENTRIES AND   0 R.H.S.

THE PROBLEM HAS  224 ROWS AND  634 COLUMNS, 1716 MATRIX ENTRIES AND   1 R.H.S.
START PHASE  CONTROL CARD

CST     B
                                             *****<[ INVERT ]>******  TIMES:   PROC. =   77.1   ELAPSED =  248.3

                  REINVERTING AFTER    0-TH ITERATION,    0 TRANSFORMATIONS WITH     0 ENTRIES.
                  NEGATIVE RIGHT HAND SIDE FROM INVERTING   -4379.252990   ROW NO.   2    ROW NAME INFEAS
                  NUMBER OF ENTRIES IN BASIS PRESENTED TO INVERSION:   381

                                             *****<[ E/INV  ]>******  TIMES:   PROC. =   90.4   ELAPSED =  280.1
COMPUTE      CONTROL CARD

CST     B
                                             *****<[ E/INP  ]>******  TIMES:   PROC. =   90.7   ELAPSED =  282.2
                                             *****<[ INVERT ]>******  TIMES:   PROC. =  166.3   ELAPSED =  473.0

                  REINVERTING AFTER   75-TH ITERATION,  297 TRANSFORMATIONS WITH   702 ENTRIES.
                  NUMBER OF ENTRIES IN BASIS PRESENTED TO INVERSION:   461

                                             *****<[ E/INV  ]>******  TIMES:   PROC. =  182.0   ELAPSED =  491.0
              MAXIMUM ERROR ON ROW   31 =4.28408@-08,   SUM 2.85043@-07
              MAXIMUM ERROR ON COL   25 =1.81899@-12,   SUM 1.27329@-11
                                             *****<[ INVERT ]>******  TIMES:   PROC. =  244.0   ELAPSED =  604.1
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ITER        OBJECTIVE FUNCT. ENTER.   MIN RED. COST  EXIT.       PIVOT RATIO  R.H.S.    ARTIFICIALS   #DJ  #TRAN

 187   1.76441972@-10  INFEAS  B42813  -1.61250000@-02  ARTFCL      35.53876017  B            28           2   851
                                              *****<[ PHASE2 ]>******  TIMES:   PROC. =  303.6   ELAPSED =   725.3
                  MAXIMUM ERROR ON ROW  92  1.26660@-07.   SUM 5.23740@-07
                  MAXIMUM ERROR ON COL  92  0.00000@+00.   SUM 0.00000@+00
                  MAXIMUM ERROR ON ROW  92  5.96046@-08.   SUM 5.06510@-07
                  MAXIMUM ERROR ON COL  12 -8.56757@-12.   SUM 2.42295@-11
                                              *****<[ BASIS  ]>******  TIMES:   PROC. =  322.5   ELAPSED =   774.3
                                              *****<[ INVERT ]>******  TIMES:   PROC. =  358.6   ELAPSED =   855.4

                        REINVERTING AFTER  225-TH ITERATION.  297 TRANSFORMATIONS WITH  1394 ENTRIES.
                        NUMBER OF ENTRIES IN BASIS PRESENTED TO INVERSION:   613

                                              *****<[ E/INV. ]>******  TIMES:   PROC. =  386.2   ELAPSED =   893.5
                                              *****<[ INVERT ]>******  TIMES:   PROC. =  447.2   ELAPSED =  1015.5

                        REINVERTING AFTER  281-TH ITERATION.  278 TRANSFORMATIONS WITH  1651 ENTRIES.
                        NUMBER OF ENTRIES IN BASIS PRESENTED TO INVERSION:   621

                                              *****<[ E/INV  ]>******  TIMES:   PROC. =  474.7   ELAPSED =  1052.5
                                              *****<[ E/PH#2 ]>******  TIMES:   PROC. =  475.6   ELAPSED =  1054.0

EXECUTION TIME =  476 SEC.    I/O TIME =  434 SEC.                          SEPT 17, 1969          TIME:  1639 HOURS
```

ITER	OBJECTIVE FUNCT.		ENTER.	MIN RED. COST	EXIT.	PIVOT RATIO	R.H.S.	ARTIFICIALS	#DJ	#TRAN
281	1.00304599E+01	CST	S1K112	0.00000000E+00	S1G112	9.17999995	B	0	0	883

ACTIVITY	CURRENT VALUE	LINE COUNT	CONSTRAINT NAME	TRANSFORMATION VECTOR	PRICING VECTOR
ARTFCL	-10.03045989	1	CST	0.00000000	1.00000000
ARTFCL	0.00000000	2	INFEAS	0.00000000	0.00000000
PFB1T1	1673.50573498	3	&PFB1T1	0.00000000	0.00000000
PFB2T1	5714.44000890	4	&PFB2T1	0.00000000	0.00000000
PFB3T1	2498.24003367	5	&PFB3T1	0.00000000	0.00000000
B2A118	256.50000000	6	&LAB1T1	0.00000000	-0.00004951
LAB2T1	1936.00000000	7	&LAB2T1	0.00000000	0.00000000
B4A319	13.19999993	8	&LAB3T1	0.00000000	-0.00005736
B1F111	49.20196968	9	-B1Z1T1	0.00000000	-0.00232000
B1F112	16.00000000	10	-B1Z2T1	0.00000000	-0.00277000
B1A118	90.00000000	11	-B1Z8T1	0.00000000	-0.00032000
B1A111	43.79803032	12	-B2Z8T1	0.00000000	-0.00032030
B3F311	29.69999993	13	-B3Z1T1	0.00000000	-0.00290000
B3A318	7.91999999	14	B3Z8T1	0.00000000	0.00000000
B4F311	75.89999962	15	-B4Z1T1	0.00000000	-0.00290000
B4Z8T1	53.61876054	16	-B4Z8T1	0.00000000	0.00000000
B4A318	133.91675978	17	-B4Z9T1	0.00000000	-0.00140000
B5F211	6.00000000	18	-B5Z1T1	0.00000000	-0.00286000
B5A218	8.00000000	19	-B5Z8T1	0.00000000	-0.00318000
B5A219	30.00000000	20	-B5Z9T1	0.00000000	-0.00494000
B6F211	37.59999990	21	-B6Z1T1	0.00000000	-0.00292000
OKS1T1	9291.78482960	22	&OKS1T1	0.00000000	0.00000000
OKS2T1	1605.87030023	23	&OKS2T1	0.00000000	0.00000000
S5K311	20.49180328	24	&OKS3T1	0.00000000	0.00000698
S6K516	1.00000000	25	&OKS5T1	0.00000000	-0.00000471
S3G115	14.24252894	26	&SGS1T1	0.00000000	-0.00000627
S5G317	2.39999999	27	&SGS3T1	0.00000000	0.00000288
LNS1T1	173.91600546	28	&LNS1T1	0.00000000	0.00000000
S3V114	1.52948738	29	&VYS1T1	0.00000000	0.00000744
VYS3T1	1106.90000367	30	&VYS3T1	0.00000000	0.00000000
HRS1T1	4035.52753967	31	&HRS1T1	0.00000000	0.00000000
HRS2T1	2855.00000715	32	&HRS2T1	0.00000000	0.00000000
HRS5T1	1684.80000208	33	&HRS5T1	0.00000000	0.00000000
S1K111	18.86999989	34	-S1Z1T1	0.00000000	-0.00213000
S1K112	9.17999995	35	-S1Z2T1	0.00000000	-0.00247000
S1K113	29.57999992	36	-S1Z3T1	0.00000000	-0.00262000
S1K114	6.63000000	37	-S1Z4T1	0.00000000	-0.00281000
S1G115	2.03999999	38	-S1Z5T1	0.00000000	-0.00262137
S1L117	8.15999997	39	-S1Z7T1	0.00000000	-0.00231000
S1H118	20.39999998	40	-S1Z8T1	0.00000000	-0.00231000
S2K111	25.29999995	41	-S2Z1T1	0.00000000	-0.00210000
S2K112	14.29999995	42	-S2Z2T1	0.00000000	-0.00244000
S2V113	19.79999995	43	-S2Z3T1	0.00000000	-0.00222364
S2K116	2.19999999	44	-S2Z6T1	0.00000000	-0.00221000
S2L117	2.19999999	45	-S2Z7T1	0.00000000	-0.00230000
S2H118	173.35999966	46	-S2Z8T1	0.00000000	-0.00246000
S3K111	24.35999990	47	-S3Z1T1	0.00000000	-0.00226000
S3V112	26.56499994	48	-S3Z2T1	0.00000000	-0.00274000
S3V113	17.32499993	49	-S3Z3T1	0.00000000	-0.00236000
S3K114	10.54551255	50	-S3Z4T1	0.00000000	-0.00336000
S3K115	7.28247104	51	-S3Z5T1	0.00000000	-0.00307000
S3K116	15.53999996	52	-S3Z6T1	0.00000000	-0.00245000
S3L117	42.00000000	53	-S3Z7T1	0.00000000	-0.00225000

S3H118	32.44499993	54	-S3Z8T1	0.00000000	-0.00231000
S3H119	5.45999999	55	-S3Z9T1	0.00000000	-0.00406000
S4K211	5.61000000	56	-S4Z1T1	0.00000000	-0.00351000
S4K212	0.94385004	57	-S4Z2T1	0.00000000	-0.00407000
S4V313	8.79999995	58	-S4Z3T1	0.00000000	-0.00408000
S4K214	2.86000000	59	-S4Z4T1	0.00000000	-0.00461000
S4K216	1.31999999	60	-S4Z6T1	0.00000000	-0.00370000
S4H218	28.59999990	61	-S4Z8T1	0.00000000	-0.00383000
S5K511	3.50819672	62	-S5Z1T1	0.00000000	-0.00605174
S5K512	6.46362718	63	-S5Z2T1	0.00000000	-0.01055174
S5V313	4.50000000	64	-S5Z3T1	0.00000000	-0.00842000
S5G315	0.50000000	65	-S5Z5T1	0.00000000	-0.01105174
S5K516	0.70000000	66	-S5Z6T1	0.00000000	-0.00755174
S5G312	1.71637275	67	-S5Z7T1	0.00000000	-0.01755174
S5H518	4.19999999	68	-S5Z8T1	0.00000000	-0.00796000
S6K512	1.00000000	69	-S6Z2T1	0.00000000	-0.00441479
S6K515	2.00000000	70	-S6Z5T1	0.00000000	-0.00641479
S4G312	3.67614995	71	-S6Z6T1	0.00000000	-0.00141479
S6H518	0.80000000	72	-S6Z8T1	0.00000000	-0.01302000
PFB1T2	1661.52986038	73	&PFB1T2	0.00000000	0.00000000
PFB2T2	1250.78000516	74	&PFB2T2	0.00000000	0.00000000
PFB3T2	1998.08005196	75	&PFB3T2	0.00000000	0.00000000
B2A128	256.78300884	76	&LAB1T2	0.00000000	-0.00003961
B5A229	27.00000000	77	&LAB2T2	0.00000000	-0.00003961
B4A329	14.39999998	78	&LAB3T2	0.00000000	-0.00004589
B1F121	49.48916091	79	-B1Z1T2	0.00000000	-0.00185600
B1F122	16.00000000	80	-B1Z2T2	0.00000000	-0.00221600
B1A128	90.00000000	81	-B1Z8T2	0.00000000	-0.00025600
B2A228	51.01699040	82	-B2Z8T2	0.00000000	-0.00025624
B3F321	32.39999998	83	-B3Z1T2	0.00000000	-0.00232000
B3A328	8.63999999	84	B3Z8T2	0.00000000	0.00000000
B4F321	82.79999924	85	-B4Z1T2	0.00000000	-0.00232000
B4Z8T2	44.39876030	86	-B4Z8T2	0.00000000	0.00000000
B4A328	131.99875973	87	-B4Z9T2	0.00000000	-0.00112000
B5F221	5.39999999	88	-B5Z1T2	0.00000000	-0.00228800
B5A228	7.19999999	89	-B5Z8T2	0.00000000	-0.00034587
B1A121	43.51083909	90	-B5Z9T2	0.00000000	-0.00175387
B6F221	47.00000000	91	-B6Z1T2	0.00000000	-0.00233600
S3K126	16.27999997	92	&OKS1T2	0.00000000	0.00000198
S4K226	1.44000000	93	&OKS2T2	0.00000000	0.00000198
S5K326	0.70700000	94	&OKS3T2	0.00000000	0.00000198
S6K526	1.00000000	95	&OKS5T2	0.00000000	-0.00000653
S3G125	5.73606515	96	&SGS1T2	0.00000000	-0.00000304
S5G327	2.42400000	97	&SGS3T2	0.00000000	-0.00000130
LNS1T2	1677.52355793	98	&LNS1T2	0.00000000	0.00000000
S3V124	0.96810274	99	&VYS1T2	0.00000000	0.00000862
VYS3T2	1038.14500943	100	&VYS3T2	0.00000000	0.00000000
S3H129	5.72000000	101	&HRS1T2	0.00000000	-0.00003744
HRS2T2	2660.00000536	102	&HRS2T2	0.00000000	0.00000000
HRS5T2	1678.24800043	103	&HRS5T2	0.00000000	0.00000000
S1K121	19.23999989	104	-S1Z1T2	0.00000000	-0.00180207
S1G122	9.35999990	105	-S1Z2T2	0.00000000	-0.00215921
S1K123	30.15999997	106	-S1Z3T2	0.00000000	-0.00219407
S1K124	6.75999999	107	-S1Z4T2	0.00000000	-0.00234607
S1G125	2.08000000	108	-S1Z5T2	0.00000000	-0.00219121
S1H127	8.31999993	109	-S1Z7T2	0.00000000	-0.00066570
S1H128	20.79999995	110	-S1Z8T2	0.00000000	-0.00018570
S2K121	23.22999990	111	-S2Z1T2	0.00000000	-0.00177708
S2K122	13.13000000	112	-S2Z2T2	0.00000000	-0.00204908
S2V123	18.17999995	113	-S2Z3T2	0.00000000	-0.00187705
S2K126	2.02000000	114	-S2Z6T2	0.00000000	-0.00186508
S2H127	2.02000000	115	-S2Z7T2	0.00000000	-0.00064717

S2H128	159.17599964	116	-S2Z8T2	0.00000000	-0.00016717
S3K221	25.51999998	117	-S3Z1T2	0.00000000	-0.00191200
S3V122	27.82999992	118	-S3Z2T2	0.00000000	-0.00229600
S3V123	18.14999998	119	-S3Z3T2	0.00000000	-0.00199200
S3K224	1.66463070	120	-S3Z4T2	0.00000000	-0.00279200
S3L125	11.13524859	121	-S3Z5T2	0.00000000	-0.00256000
S3K125	5.67868621	122	-S3Z6T2	0.00000000	-0.00206400
S3L127	7.47281573	123	-S3Z7T2	0.00000000	-0.00180000
S3H128	33.98999989	124	-S3Z8T2	0.00000000	-0.00015200
S3H127	36.52718427	125	-S3Z9T2	0.00000000	-0.00155200
S4K321	2.87380023	126	-S4Z1T2	0.00000000	-0.00297270
S4K222	5.03999999	127	-S4Z2T2	0.00000000	-0.00342070
S4V323	9.59999990	128	-S4Z3T2	0.00000000	-0.00326400
S4K224	3.11999999	129	-S4Z4T2	0.00000000	-0.00385270
S4K221	3.24619976	130	-S4Z6T2	0.00000000	-0.00312470
S4H228	31.19999993	131	-S4Z8T2	0.00000000	-0.00306400
S5K521	6.41738970	132	-S5Z1T2	0.00000000	-0.00440122
S5K522	3.76943420	133	-S5Z2T2	0.00000000	-0.00800122
S5V323	4.54499999	134	-S5Z3T2	0.00000000	-0.00673600
S5K525	0.50500000	135	-S5Z5T2	0.00000000	-0.00800122
S5K321	17.82261019	136	-S5Z6T2	0.00000000	-0.00560122
S5G322	4.51256574	137	-S5Z7T2	0.00000000	-0.01360122
S5H528	4.24200000	138	-S5Z8T2	0.00000000	-0.00636800
S6K522	1.00000000	139	-S6Z2T2	0.00000000	-0.00297835
S6K525	2.00000000	140	-S6Z5T2	0.00000000	-0.00457835
S3K124	10.01726654	141	-S6Z6T2	0.00000000	-0.00057835
S6H528	0.80000000	142	-S6Z8T2	0.00000000	-0.01041600
B1F138	3.35270793	143	&PFB1T3	0.00000000	0.00003125
B6F231	65.79999924	144	&PFB2T3	0.00000000	0.00003125
PFB3T3	1568.25502169	145	&PFB3T3	0.00000000	0.00000000
B2A138	302.96407699	146	&LAB1T3	0.00000000	0.00006641
B5A239	24.00000000	147	&LAB2T3	0.00000000	0.00006641
B4A339	15.59999990	148	&LAB3T3	0.00000000	-0.00004016
B1F231	7.01870617	149	-B1Z1T3	0.00000000	-0.00292712
B1F132	16.00000000	150	-B1Z2T3	0.00000000	-0.00324212
B1F131	85.98129383	151	-B1Z8T3	0.00000000	-0.00432712
B2A238	56.13592244	152	-B2Z8T3	0.00000000	-0.00438797
B3F331	33.75000000	153	-B3Z1T3	0.00000000	-0.00203000
B3A338	9.00000000	154	B3Z8T3	0.00000000	0.00000000
B4F331	89.69999981	155	-B4Z1T3	0.00000000	-0.00203000
B4Z8T3	35.53876017	156	-B4Z8T3	0.00000000	0.00000000
B4A338	130.43875979	157	-B4Z9T3	0.00000000	-0.00098000
B5F231	4.80000000	158	-B5Z1T3	0.00000000	-0.00360512
B5A238	6.39999999	159	-B5Z8T3	0.00000000	-0.00591159
IPF3T2	1.00000000	160	-B5Z9T3	0.00000000	-0.00591159
B1A138	86.64729207	161	-B6Z1T3	0.00000000	-0.00368775
S3K134	13.79999995	162	&OKS1T3	0.00000000	0.00007352
S3K236	17.75999999	163	&OKS2T3	0.00000000	0.00007352
S5K331	17.52332087	164	&OKS3T3	0.00000000	0.00000861
S6K536	0.80000000	165	&OKS5T3	0.00000000	-0.00000043
S1G133	8.23859294	166	&SGS1T3	0.00000000	0.00006999
S5G337	2.42400000	167	&SGS3T3	0.00000000	0.00000575
LNS1T3	1316.43200649	168	&LNS1T3	0.00000000	0.00000000
S3V133	19.79999995	169	&VYS1T3	0.00000000	0.00010579
S5V333	4.54499999	170	&VYS3T3	0.00000000	-0.00000041
S3H139	6.23999999	171	&HRS1T3	0.00000000	-0.00003580
S4H238	37.67466670	172	&HRS2T3	0.00000000	-0.00003575
HRS5T3	1710.24800046	173	&HRS5T3	0.00000000	0.00000000
S1K131	19.60999990	174	-S1Z1T3	0.00000000	-0.00513745
S1G132	9.53999996	175	-S1Z2T3	0.00000000	-0.00534745
S1K133	22.50140696	176	-S1Z3T3	0.00000000	-0.00548045
S1K134	6.88999999	177	-S1Z4T3	0.00000000	-0.00561345

S1L135	2.11999999	178	-S1Z5T3	0.00000000	-0.00200900
S1H137	8.47999990	179	-S1Z7T3	0.00000000	-0.00044746
S1H138	21.19999993	180	-S1Z8T3	0.00000000	-0.00002746
S2K131	23.45999992	181	-S2Z1T3	0.00000000	-0.00507969
S2K132	13.25999999	182	-S2Z2T3	0.00000000	-0.00531769
S2V133	18.35999990	183	-S2Z3T3	0.00000000	-0.00525825
S2K136	2.03999999	184	-S2Z6T3	0.00000000	-0.00515669
S2H137	2.03999999	185	-S2Z7T3	0.00000000	-0.00042000
S2Z8T3	28.88808399	186	-S2Z8T3	0.00000000	0.00000000
S3K231	27.83999991	187	-S3Z1T3	0.00000000	-0.00544900
S3V132	26.97825658	188	-S3Z2T3	0.00000000	-0.00584100
S3K232	1.92851729	189	-S3Z3T3	0.00000000	-0.00557500
S3K132	1.45322603	190	-S3Z4T3	0.00000000	-0.00621900
S3L135	24.59999990	191	-S3Z5T3	0.00000000	-0.00224000
IOK1T2	1.00000000	192	-S3Z6T3	0.00000000	-0.00558200
S3H237	48.00000000	193	-S3Z7T3	0.00000000	-0.00069768
S3H138	37.07999992	194	-S3Z8T3	0.00000000	0.00000476
S2H138	189.64008384	195	-S3Z9T3	0.00000000	-0.00122024
S4V331	2.28240267	196	-S4Z1T3	0.00000000	-0.00317458
S4V332	5.45999999	197	-S4Z2T3	0.00000000	-0.00309058
S4V333	10.39999998	198	-S4Z3T3	0.00000000	-0.00282458
S4V334	3.38300000	199	-S4Z4T3	0.00000000	-0.00352458
S4V336	1.55999999	200	-S4Z6T3	0.00000000	-0.00317458
S4Z8T3	3.87466674	201	-S4Z8T3	0.00000000	0.00000000
S5K531	6.71667902	202	-S5Z1T3	0.00000000	-0.00469096
S5K532	3.76943420	203	-S5Z2T3	0.00000000	-0.00784096
ISG1T1	0.66149559	204	-S5Z3T3	0.00000000	-0.00582912
S5K535	0.50500000	205	-S5Z5T3	0.00000000	-0.00784096
S5K536	0.70700000	206	-S5Z6T3	0.00000000	-0.00574096
S5G332	4.51256574	207	-S5Z7T3	0.00000000	-0.01274096
S5H538	4.24200000	208	-S5Z8T3	0.00000000	-0.00557200
S6K532	0.80000000	209	-S6Z2T3	0.00000000	-0.00366215
S6K535	1.59999999	210	-S6Z5T3	0.00000000	-0.00506215
S4K331	4.34759732	211	-S6Z6T3	0.00000000	-0.00156215
S6H538	0.64000000	212	-S6Z8T3	0.00000000	-0.00911400
RISG11	0.33850441	213	+RISG11	0.00000000	0.00000000
RIPF11	0.04905348	214	+RIPF11	0.00000000	0.00000000
IHR1T1	1.00000000	215	+RIHR11	0.00000000	1.04929782
RILA11	1.00000000	216	+RILA11	0.00000000	0.00000000
ILA2T1	1.00000000	217	RILA21	0.00000000	-2.07364341
IOK2T1	0.48640945	218	+RIOK12	0.00000000	0.24505703
RISG22	1.00000000	219	+RISG22	0.00000000	0.00000000
IHR2T1	1.00000000	220	RIHR21	0.00000000	-0.88936666
RIOK21	0.51359055	221	+RIOK21	0.00000000	0.00000000
RIPF21	1.00000000	222	+RIPF21	0.00000000	0.00000000
ILA3T1	1.00000000	223	+RIPF32	0.00000000	0.02500000
IPF1T1	0.95094652	224	+RILA31	0.00000000	0.89279557

VARIABLE NAME	VALUE	LINE COUNT	REDUCED COST	POSITION IN BASIS
B1A111	43.79803032	1	0.00000000	12
B1A118	90.00000000	2	0.00000000	11
B1A211	0.00000000	3	0.00201000	0
B1A218	0.00000000	4	0.00201000	0
B1F111	49.20196968	5	0.00000000	9
B1F112	16.00000000	6	0.00000000	10
B1F118	0.00000000	7	0.00400000	0
B1F211	0.00000000	8	0.00000000	0
B1F212	0.00000000	9	0.00000000	0
B2A118	256.50000000	10	0.00000000	6
B2A218	0.00000000	11	0.00203970	0

```
         RIPF21              1.00000000        632          0.00000000          222
         RIPF32              0.00000000        633          0.02500000            0
         RILA31              0.00000000        634          0.89279557            0
              MAXIMUM ERROR ON ROW  98 -4.27477@-07,   SUM 1.48900@-06
              MAXIMUM ERROR ON COL 215  1.45519@-11,   SUM 4.56666@-11
                                                  *****<[ BASIS  ]>****** TIMES:   PROC. =  501.4   ELAPSED =  1090.1
                                                  *****<[ INPUT  ]>****** TIMES:   PROC. =  503.0   ELAPSED =  1091.9
COST RANGING CONTROL CARD

                                S-E-N-S-I-T-I-V-I-T-Y   A-N-A-L-Y-S-I-S

  VARIABLE NAME  BASIS STATUS      ACTUAL COST      LOWER RANGE  ENTERING  EXITING    UPPER RANGE  ENTERING  EXITING.
       ISG1T1         IN            -0.17000          -0.26636   S4G331    S4K331       -0.17141   OKS5T3    S4K212
IPF1T1
       IPF1T1         IN            -0.25000          -0.30000   RIPF32    RIPF11       -0.16667   IPF2T1    IPF1T1
IHR1T1
       IHR1T1         IN            -0.50000        OPEN                                 0.54930   RIHR11    S3L127
ILA1T1
       ILA1T1                        0.50000          -0.09083   ILA1T1    81F138      OPEN
IOK1T2
       IOK1T2         IN            -1.00000        OPEN                                -0.75494   RIOK12    S3L125
ILA2T1
       ILA2T1         IN             1.50000        OPEN                               OPEN
ISG2T2
       ISG2T2                        0.50000           0.16737   ISG2T2    S3K125      OPEN
IHR2T1
       IHR2T1         IN             0.80000        OPEN                               OPEN
IOK2T1
       IOK2T1         IN            -0.75000          -0.75362   VY53T3    S4K212       -0.43216   S4K231    S3K224
IPF2T1
       IPF2T1                       -0.25000          -0.37500   IPF2T1    IPF1T1      OPEN
IPF3T2
       IPF3T2         IN            -0.15000        OPEN                                -0.12500   RIPF32    RIPF11
ILA3T1
       ILA3T1         IN            -1.00000        OPEN                                -0.10720   RILA31    81F138
END
** - SOLUTION IS UNBOUNDED FOR PARAMETER VALUES ≥ PRESENT PARAMETER

CONCLUDE     CONTROL CARD

END OF RUN MIXED
```

APPENDIX E

MIXED ALGOL PROGRAM

```
        BURROUGHS B-5500 ALGOL COMPILER LEVEL   4    SATURDAY,    8/30/69,  9:40 AM.

    BEGIN                                                                  00000100    0000
                                                     START OF SEGMENT **********    2
    FILE IN  NPUT DISK SERIAL (2,10,30);                                   00000200    0000
    FILE OUT OUTPUT DISK SERIAL [20:12] (2,30,30,SAVE 14);                 00000300    0003
    FILE IN INPUT2 DISK SERIAL (2,10,30);                                  00000400    0007
    FORMAT FM1(10A6), FM2(A6,5(A6,A6)),FM3(A6),                            00000500    0010
                                                     START OF SEGMENT **********    3
    FM4(2A6),FM5(3A6,F12.6),FM6(7F10.2),                                   00000600    0010
    FM7(A6,X6,A1,A1,A3,A1,F12.6);                                          00000700    0010
                                                  3 IS   39 LONG, NEXT SEG    2
    ARRAY FC[0:12,0:12],INV[0:10];                                         00000800    0010
    ALPHA NAM, NOM;                                                        00000900    0014
    ALPHA ARRAY A,SIG[0:10];                                               00001000    0014
    REAL  D1,D2,D3;                                                        00001100    0016
    REAL X;                                                                00001200    0016
    INTEGER I,J,L,LL,NTP;                                                  00001300    0016
    LIST LA(FOR L:=1 STEP 1 UNTIL LL DO[A[L]]),                            00001400    0016
         LAX(LA,X),                                                        00001500    0026
         LB(NAM,FOR L:=1 STEP 1 UNTIL 5 DO[SIG[L],A[L]]),                  00001600    0037
         LC(J,FOR L:=1 STEP 1 UNTIL NTP DO[FC[J,L]]);                      00001700    0051
                                                                           00001800    0064
    LABEL L1,L2,L3,L4,L5,L6,L7,FIN;                                        00001900    0064
    LABEL L8,L9,L10,L11,L12;                                               00002000    0064
    LL:=4;                                                                 00002100    0064
    NTP:=3;                                                                00002200    0065
    READ(INPJT,/,D1,D2,D3);                                                00002300    0066
    FOR J:=1 STEP 1 UNTIL 12 DO READ(INPUT,/,LC);                          00002400    0078
    L1:                                                                    00002500    0085
    READ(INPUT2,FM1,LA);                                                   00002600    0086
    WRITE(OUTPUT,FM1,LA);                                                  00002700    0089
    IF A[1] EQL "HEADIN" THEN                                              00002800    0093
    BEGIN LL:=10;                                                          00002900    0094
    READ(INPUT2,FM1,LA);                                                   00003000    0095
    WRITE(OUTPUT,FM1,LA);                                                  00003100    0099
    WRITE(OUTPUT,FM6,D1,D2,D3);                                            00003200    0102
    FOR J:=1 STEP 1 UNTIL 12 DO WRITE(OUTPUT,FM6,LC);                      00003300    0116
    READ(INPUT2,FM3,NAM);                                                  00003400    0122
    WRITE(OUTPUT,FM3,NAM);                                                 00003500    0130
    LL:=4;                                                                 00003600    0138
    END ELSE                                                               00003700    0139
    IF A[1] EQL "ROW 10" THEN GO TO L2;                                    00003800    0139
    GO TO L1;                                                              00003900    0141
    L2:                                                                    00004000    0143
    READ(INPUT2,FM2,LB);                                                   00004100    0143
    IF NAM NEQ "END   " THEN                                               00004200    0146
    BEGIN  WRITE(OUTPUT,FM2,LB);                                           00004300    0147
    GO TO L2;                                                              00004400    0151
    END ELSE                                                               00004500    0153
    BEGIN                                                                  00004600    0153
    NOM:=NAM;                                                              00004700    0153
    READ(INPUT,FM3,NAM);                                                   00004800    0154
                                                                           00004900    0162
                                                                           00005000    0162
    IF NAM NEQ "NEWRWS" THEN GO TO L12;                                    00005100    0162
    L11:                                                                   00005200    0164
    READ(INPUT,FM2,LB);                                                    00005300    0164
    IF NAM EQL "ENDRWS" THEN GO TO L12;                                    00005400    0167
    WRITE(OUTPUT,FM2,LB);                                                  00005500    0169
    GO TO L11;                                                             00005600    0172
    L12:                                                                   00005700    0175
```

```
WRITE(OUTPUT,FM3,NOM);                                          0175
READ(INPUT2,FM3,NAM,NOM);                                       0182
WRITE(OUTPUT,FM3,NAM,NOM);                                      0193
GO TO L3;                                                       0204
END;                                                            0204
L3:                                                             0204
READ(INPUT2,FM5,A[1],A[2],A[3],X);                              0205
IF A[1] EQL "FIRST " THEN                                       0219
BEGIN LL:=5;                                                    0220
READ(INPUT,FM3,NAM);                                            0222
IF NAM EQL "NEWCOL" THEN                                        0230
BEGIN                                                           0231
L9:                                                             0232
READ(INPUT,FM5,NAM,SIG[1],SIG[2],X);                            0232
IF NAM NEQ "ENDCOL" THEN BEGIN                                  0246
WRITE(OUTPUT,FM5,NAM,SIG[1],SIG[2],X);                          0248
GO TO L9;                                                       0262
END; END;                                                       0262
WRITE(OUTPUT,FM5,A[1],A[2]);                                    0262
L7:                                                             0274
READ(INPUT2,FM7,LAX);                                           0275
IF A[2] EQL "B" THEN BEGIN                                      0278
I:=0; GO TO L8;                                                 0280
END ELSE                                                        0281
IF A[2] EQL "S" THEN BEGIN                                      0281
I:=6; GO TO L8;                                                 0283
END ELSE                                                        0284
IF A[1] EQL "EOF    " THEN                                      0284
BEGIN                                                           0286
READ(INPUT,FM3,NAM);                                            0286
IF NAM EQL "NEWBRW" THEN BEGIN                                  0295
COMMENT;                                                        0297
L10: READ(INPUT,FM7,LAX);                                       0297
IF A[1] NEQ "ENDBRW" THEN                                       0300
BEGIN WRITE(OUTPUT,FM7,LAX);                                    0301
GO TO L10;                                                      0305
END ELSE A[1]:="EOF    " END; GO TO L4;                         0308
END ELSE                                                        0312
BEGIN WRITE(OUTPUT,FM7,LAX);                                    0312
GO TO L7 END;                                                   0315
L8: J:=0;                                                       0316
IF A[3] EQL "1" THEN J:=1; IF A[3] EQL "2" THEN J:=2;           0317
IF A[3] EQL "3" THEN J:=3; IF A[3] EQL "4" THEN J:=4;           0322
IF A[3] EQL "5" THEN J:=5; IF A[3] EQL "6" THEN J:=6;           0326
COMMENT;IF A[5] EQL "1" THEN  L:=1                              0331
ELSE IF A[5] EQL "2" THEN L:=2                                  0332
ELSE IF A[5] EQL "3" THEN L:=3                                  0335
ELSE L:=0;                                                      0338
I:=I+J;                                                         0340
X:=X×FC[I,L];                                                   0341
WRITE(OUTPUT,FM7,LAX);                                          0343
GO TO L7;                                                       0347
END;                                                            0347
IF A[3] EQL "COST1 " THEN BEGIN                                 0347
IF X GTR-1 THEN X :=X×1,@-6;                                    0349
X:=X×D1;A[3]:="CST    ";                                        0351
END ELSE                                                        0354
IF A[3] EQL "COST2 " THEN BEGIN                                 0354
IF X GTR-1 THEN X:=X×1,@-6;                                     0359
X:=X×D2;A[3]:="CST    ";                                        0362
END ELSE                                                        0364
IF A[3] EQL "COST3 " THEN BEGIN                                 0364
```

```
        IF X GTR 1 THEN X:=X×1,@-6J                                00012000   0369
        X:=X×D3J A[3]:="CST    "J                                  00012100   0372
        ENDJ                                                       00012200   0374
        WRITE(OUTPUT,FM5,A[1],A[2],A[3],X);                        00012300   0374
        GO TO L3J                                                  00012400   0392
        L4:                                                        00012500   0392
        WRITE(OUTPUT,FM3,A[1]]J                                    00012600   0393
        READ(INPUT,FM3,NAM)J                                       00012700   0402
        IF NAM EQL "CHANGE" THEN BEGIN                             00012800   0410
        WRITE(OUTPUT,FM3,NAM)J                                     00012900   0412
        L6:                                                        00013000   0420
        READ(INPUT,FM5,NAM,A[1],A[2],X)J                           00013100   0421
        IF NAM EQL "END    " THEN                                  00013200   0434
        BEGIN  WRITE(OUTPUT,FM3,NAM)J                              00013300   0435
        GO TO L5J                                                  00013400   0444
        END ELSE  WRITE(OUTPUT,FM5,NAM,A[1],A[2],X)J               00013500   0444
        GO TO L6J                                                  00013600   0459
        ENDJ                                                       00013700   0459
        L5:                                                        00013800   0459
        READ(INPUT2,FM1,LA)[FIN]J                                  00013900   0460
        WRITE(OUTPUT,FM1,LA)J                                      00014000   0464
        GO TO L5J                                                  00014100   0468
        FIN:                                                       00014200   0468
        LOCK(OUTPUT)J                                              00014300   0469
        END.                                                       00014400   0470
                                                      2 IS  474 LONG, NEXT SEG    1
     OUTPUT(W) IS SEGMENT NUMBER 0004,PRT ADDRESS IS 0071
     BLOCK CONTROL IS SEGMENT NUMBER 0005,PRT ADDRESS IS 0005
     INPUT(W) IS SEGMENT NUMBER 0006,PRT ADDRESS IS 0070
     GO TO SOLVER  IS SEGMENT NUMBER 0007,PRT ADDRESS IS 0167
     ALGOL WRITE   IS SEGMENT NUMBER 0008,PRT ADDRESS IS 0014
     ALGOL READ    IS SEGMENT NUMBER 0009,PRT ADDRESS IS 0015
     ALGOL SELECT  IS SEGMENT NUMBER 0010,PRT ADDRESS IS 0016
                                                      1 IS    2 LONG, NEXT SEG    0
                                                     11 IS   69 LONG, NEXT SEG    0
NUMBER OF SYNTAX ERRORS DETECTED =   0. NUMBER OF SEQUENCE ERRORS DETECTED =    0
COMPILER TIMES: PROCESSOR =   10 SECONDSJ IO =   3 SECONDSJ ELAPSED =   16 SECONDS.
PRT SIZE = 120J TOTAL SEGMENT SIZE =   584 WORDSJ DISK SIZE =  29 SEGSJ NO. PGM. SEGS =  11
ESTIMATED CORE STORAGE REQUIREMENT =  3402 WORDS.
```

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He attended elementary school at the Instituto Mexico and Junior High School at Colegio San José Insurgentes, both in Mexico City. In 1953 he initiated his preparatory schooling at Instituto Tecnologico de Monterrey and in 1955 initiated his college studies at the same Institute.

He obtained a Bachelor's degree in Chemical and Administrative Engineering in 1960.

During 1960 and 1961 he worked as research assistant at the Instituto de Investigaciones Industriales in Monterrey, Mexico.

In 1962 he was awarded a UNESCO fellowship to begin graduate studies at Pennsylvania State University. At the end of this year he finished his course work and in 1968 he finished his Master's thesis, "A Contribution to the Theory of Conveyors."

In 1963 he became an assistant professor in the Industrial Engineering Department at the Instituto Tecnologico de Monterrey, Mexico, and in 1964 he was named Head of the Department.

He obtained a grant from the Ford Foundation in 1965 to attend the Summer Institute on Use of Computers in Engineering Education at the University of Michigan.

In 1965 he married the former Maria Beatriz Garza and in 1966 their first son, Gonzalo, Jr., was born in Monterrey, Mexico. A second son, Luis Alberto, was born in Atlanta, Georgia, in 1968.

He began working toward a Ph.D. degree in Industrial Engineering at the Georgia Institute of Technology in September, 1966.

He is member of the following societies: Alpha Pi Mu, Sigma Xi, Operations Research Society of America, and The Institute of Management Sciences.